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Abstract

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ON ONE CLASS OF ULTRAPARABOLIC EQUATIONS

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If in a second-order partial differential equation the matrix of the coefficients of the second derivatives is nonnegative definite, and its rank is less than $N-1$, where N is the number of independent variables, then it is natural to call this equation ultraparabolic. Interest in such equations arose quite long ago. It suffices to point out that such an equation describes the process of Brownian motion in phase space ([1]). Boundary-value problems for ultraparabolic, or, as they are also called, elliptic-parabolic equations were studied in the works ([2,3]).

It is known that solutions of a parabolic equation possess a number of properties closely connected with one another:

1. If a solution assumes its maximum value inside the domain of definition, then there exists a subdomain where the solution is equal to this maximum value (the strengthened maximum principle).
2. If the coefficients of the equation are sufficiently smooth, then the solution and all its derivatives up to a sufficiently high order can be estimated in any domain in terms of the maximum of the modulus of the solution in a somewhat larger domain (interior a priori estimates).
3. There exists a fundamental solution of the Cauchy problem, which has a singularity only at one point on the initial plane and is a smooth positive function for the remaining values of the arguments.

Let us consider the ultraparabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(x,y,t)\frac{\partial^2 u}{\partial y_i \partial y_j} + \sum_{i=1}^{n} a_i(x,y,t)\frac{\partial u}{\partial y_i} +$$

$$+a(x,y,t)u + \sum_{i=1}^{m} b_i(x,y,t)\frac{\partial u}{\partial x_i}, \tag{1}$$

where $x=(x_1,x_2,\dots,x_m)$, $y=(y_1,y_2,\dots,y_n)$, and the matrix $\|a_{ij}\|$ is positive definite. It is easy to see that the properties listed above do not, generally speaking, hold for solutions of equation (1). If, for example, the coefficients a_{ij}, a_i, a do not depend on x, and b_i do not depend on y, then equation (1) is easily reduced to an ordinary parabolic equation, and its fundamental solution has a δ-type singularity along the characteristic of the first-order equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{m} b_i(x,t) \frac{\partial u}{\partial x_i}.$$

A similar situation in a considerably more general form is investigated in the work ([4]).

Here we shall examine the case when the rank of the matrix $\|\partial b_i / \partial y_j\|$ is equal to m. It follows from the results of ([5]) that in this case property 1 holds. We shall show that, under additional restrictions, such an equation has a fundamental solution and that a priori estimates are valid.

Theorem 1. Suppose that in equation (1) $m \le n$,

$$0 < \mu_1 < \left| \frac{D(b_1, \dots, b_m)}{D(y_1, \dots, y_m)} \right| < \mu_2, \qquad \sum_{i,j=1}^{n} a_{ij}(x,y,t)\xi_i \xi_j \ge \mu_1 \sum_{i=1}^{n} \xi_i^2$$

in the domain $H\{-\infty < x_i < \infty;\ -\infty < y_i < \infty;\ 0 \leqslant t \leqslant T\}$ for all real ξ_i, where μ_1, μ_2 and T are constants. Let the coefficients a_{ij}, a_i, a, their derivatives up to the second order inclusive, and the derivatives of the coefficients b_i up to the fourth order inclusive be bounded in the domain H.

Then in the domain H there exists a unique fundamental solution $G(x, y, t, \xi, \eta, \tau)$ of equation (1), possessing for $t > \tau$ continuous derivatives entering into equation (1). The Cauchy problem for equation (1) with initial condition

$$u(x, y, \tau) = \varphi(x, y) \tag{2}$$

has the unique bounded solution

$$u(x,y,t) = \iint G(x,y,t,\xi,\eta,\tau)\varphi(\xi,\eta)\, d\xi\, d\eta, \tag{3}$$

if $\varphi(x, y)$ is bounded and continuous.

For the proof, we shall simplify equation (1) by making the change of variables $t' = t;\ x_i' = x_i;\ y_i' = b_i(x, y, t)$. We shall consider only the case $m = n$; the case $m < n$ is treated analogously. Thus the problem is reduced to the construction of the fundamental solution of the equation

$$\mathcal{L}u \equiv -\frac{\partial u}{\partial t} + \sum_{i,j=1}^{n} a_{ij}(x,y,t)\frac{\partial^2 u}{\partial y_i \partial y_j} + \sum_{i=1}^{n} a_i(x,y,t)\frac{\partial u}{\partial y_i} + a(x,y,t)u + \sum_{i=1}^{n} y_i \frac{\partial u}{\partial x_i}, \tag{4}$$

where the coefficients a_{ij}, a_i, and a are bounded in the domain H, together with derivatives up to the second order inclusive.

Denote by $W_S(P,Q)$ the fundamental solution of equation (4), if in it one sets $a_{ij} = a_{ij}(S)$, where the point $S(\xi,\sigma,\theta)$ is fixed, and $a_i \equiv a \equiv 0$; here P is the point (x,y,t), Q is (ξ,η,τ). This solution is easily obtained by applying the Fourier transform:

$$W_S(P,Q) = \left(\frac{\sqrt{3}}{2\pi(t-\tau)^2}\right)^n \det A \cdot \exp\left\{-\frac{(A(y-\eta),\, y-\eta)}{4(t-\tau)} - \frac{3}{(t-\tau)^3}\left(A\left(x-\xi+\frac{y+\eta}{2}(t-\tau)\right),\, x-\xi+ \frac{y}{}\right.\right.$$

where the matrix A is inverse to the matrix $\|a_{ij}(S)\|$.

Following Levi, we shall seek the fundamental solution for equation (4) in the form

$$G(P,Q) = W_Q(P,Q) + \int_\tau^t \iiint W_S(P,S)\Phi(S,Q)\,d\xi\,d\sigma\,d\theta. \tag{5}$$

Denote the integral appearing on the right-hand side of this equality by $I(P,Q)$. Assuming that

$$\frac{\partial I}{\partial y_i} = \int_\tau^t \iiint \frac{\partial W_S(P,S)}{\partial y_i}\Phi(S,Q)\,d\xi\,d\sigma\,d\theta,$$

$$\frac{\partial^2 I}{\partial y_i \partial y_j} = \int_\tau^t \iiint \frac{\partial^2 W_S(P,S)}{\partial y_i \partial y_j}\Phi(S,Q)\,d\xi\,d\sigma\,d\theta; \tag{6}$$

$$\frac{\partial I}{\partial t} = \Phi(P,Q) + \int_\tau^t \iiint \frac{\partial W_S(P,S)}{\partial t}\Phi(S,Q)\,d\xi\,d\sigma\,d\theta,$$

$$\frac{\partial I}{\partial x_i} = \int_\tau^t \iiint \frac{\partial W_S(P,S)}{\partial x_i}\Phi(S,Q)\,d\xi\,d\sigma\,d\theta, \tag{7}$$

and substituting expression (5) into equation (4), we obtain for the function $\Phi(P,Q)$ the integral equation

$$\Phi(P,Q) = K(P,Q) + \int_\tau^t \iint K(P,S)\Phi(S,Q)\, d\xi\, d\sigma\, d\theta, \tag{8}$$

where

$$K(P,Q) = \mathcal{L}_P W_Q(P,Q).$$

The solution of equation (8) can be obtained by the method of successive approximations. In this case the function $\Phi(P,Q)$ satisfies the inequality

$$|\Phi(P,Q)| \le M\big((t-\tau)^{-1/2} + e^{M|\eta|}\big) Z(\mu_3, P, Q), \tag{9}$$

where

$$Z(\mu,P,Q) = \frac{1}{(t-\tau)^{2n}} \exp\left\{ -\mu \frac{|y-\eta|^2}{t-\tau} - \frac{\mu}{(t-\tau)^3} \left| x - \xi + \frac{y+\eta}{2}(t-\tau)^2 \right| \right\},$$

and M and μ are constants. With the aid of this estimate and the integral equation (8), one can show that, for the constructed function $\Phi(P,Q)$, the equalities (6), (7) hold. Thus the existence of a fundamental solution is proved, which, according to (5) and (9), satisfies the inequalities

$$|G(P,Q)| \le M_1 Z(\mu_4, P, Q)\{1+(t-\tau)e^{M|\eta|}\} \le\le M_2 Z(\mu_5, P, Q)\{1+(t-\tau)e^{M|y|}\}. \tag{10}$$

Formula (3) gives the solution of the Cauchy problem (4), (2), and this solution grows no faster than $e^{M|y|}$. It is easy to prove uniqueness of the solution of the Cauchy problem in the class of functions growing no faster than $\exp\{M(|x|^2 + |y|^2)\}$. This is achieved in the usual way, by introducing the auxiliary function $\exp\{\alpha(|x|^2 + |y|^2)e^{\beta t} + \gamma t\}$ with a suitable choice of the constants α, β, γ. Thus it is proved that the solution of problem (4), (2) is bounded and $|u(x,y,t)| \le \sup |\varphi(x,y)|$. From the maximum principle there also follows the positivity of the function $G(P,Q)$ for $t > \tau$.

The greatest difficulty in the proof of Theorem 1 is the need to verify the validity of equality (7). The natural estimate

$$\left| \frac{\partial W_S(P,S)}{\partial x_i} \right| \le \frac{M}{(t-\tau)^{3/2}} Z(\mu, P, S)$$

does not allow one to prove uniform convergence of the integral (7). Nor does the usually applied extraction of the principal part of the integrand help. Relation (7) can be obtained only by studying integrals of several first terms of the expansion of the integrand functions. This same circumstance is the principal obstacle to the proof of Theorem 1 under the natural assumptions: all coefficients belong to the class C_α, and $b_i \in C_{1+\alpha}$.

The constructed fundamental solution makes it possible to obtain interior a priori estimates for the solution of equation (1).

Let Ω be a domain in the space $(x_1, \dots, x_m, y_1, \dots, y_n)$; let D be the cylinder $\Omega \times (\tau, T)$, and let D_δ be the subdomain of the cylinder D separated from its base and lateral boundary by more than δ.

Theorem 2. *Suppose that the coefficients of equation (1) satisfy the conditions of Theorem 1, and that the function $u(x,y,t)$ is a continuous solution of equation (1) in $\overline{D}$, having continuous second derivatives in D. Then the derivative functions of $u(x,y,t)$ entering equation (1) are bounded in the cylinder D_δ by a constant depending only on the coefficients of the equation,*

$$\max_{\overline{D}} |u(x,y,t)|$$

and δ.

As above, instead of equation (1) one may consider equation (4)

For the proof, first of all we note that the function

$$v(P) = \int_\tau^t \iint G(P,S) f(S)\, d\xi\, d\sigma\, d\theta + \iint G(P,Q)\varphi(\xi,\eta)\, d\xi\, d\eta$$

is a solution of the Cauchy problem for the inhomogeneous equation $\mathcal{L}v = -f(x,y,t)$ with condition (2), if the function $f(x,y,t)$ is bounded in $\bar{D}$ together with its first derivatives. Moreover, the functions $\frac{\partial G}{\partial \eta}(P,Q)$ and $\frac{\partial^2 G}{\partial y\, \partial \eta}(P,Q)$ are bounded and continuous everywhere for $|P-Q| > \delta_1 > 0$.

Let $\omega(x,y,t)$ be a smooth function equal to one in $D_{\delta/2}$ and to zero outside $D_{\delta/4}$. If $u(x,y,t)$ is a solution of equation (4), then

$$\mathcal{L}(\omega u) = \omega \mathcal{L} u + u \mathcal{L}\omega + 2\sum_{i,j=1}^n \frac{\partial u}{\partial y_i}\frac{\partial \omega}{\partial y_j} - au\omega$$

$$= u\psi(x,y,t) + \sum_{i=1}^n \psi_i(x,y,t)\frac{\partial u}{\partial y_i}.$$

Here ψ and ψ_i are smooth functions that are nonzero only in $D_{\delta/4} \setminus D_{\delta/2}$. Consequently, in the domain D_δ

$$u(P) = \omega u(P) = -\int_\tau^t \iint G(P,S)\left[u(S)\psi(S) + \sum_{i=1}^n \psi_i(S)\frac{\partial u(S)}{\partial \sigma_i}\right] d\xi\, d\sigma\, d\theta$$

$$= \int_\tau^t \iint \left[-G(P,S)\psi(S) + \sum_{i=1}^{n} \frac{\partial(G(P,S)\psi_i(S))}{\partial\sigma_i} \right] u(S)\, d\xi\, d\sigma\, d\theta. \qquad (11)$$

From this follows the estimate for $\partial u/\partial y_i$ in the domain D_δ. Taking this estimate into account for the domain $D_{\delta/4}$ and applying representation (11) once again, we obtain the assertion of the theorem.

In conclusion, let us note that the fundamental solution of equation (1) is the transition probability density of the Markov diffusion process associated with this equation. The existence and continuity of such a density were proved (also for a more general process) earlier ([6], [7]). Here it has been established that this density is a classical solution of equation (1).

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Note: Figure translations are in progress. See original paper for figures.