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Abstract

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ON THE PRINCIPLE OF LIMITING AMPLITUDE

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Let S be a closed smooth star-shaped surface in three-dimensional space, and let D be a domain of three-dimensional space containing infinity, whose boundary is the surface S. Denote by $u(x,t)$, $x=(x_1,x_2,x_3)$, the solution in D of the wave equation

$$-\frac{\partial^2 u}{\partial t^2}+\Delta u=f(x)e^{i\omega t}, \tag{1}$$

satisfying the initial conditions

$$u(x,t)\big|_{t=0}=\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=0 \tag{2}$$

and one of the boundary conditions on S:

$$u(x,t)\big|_S=0, \tag{3}$$

$$\left.\frac{\partial u(x,t)}{\partial n}\right|_S=0, \tag{4}$$

$$\left(\frac{\partial u(x,t)}{\partial n}+\sigma(x)u\right)\Big|_S=0, \tag{5}$$

where n is the normal to the surface S exterior with respect to D, and $\sigma(x)>0$ is a smooth function on the boundary S. The function $f(x)$ in (1) is assumed smooth and finite, with support in some domain Ω, which may be regarded as a ball of sufficiently large radius. In addition, we shall assume the function $f(x)$ to be equal to zero on S, together with its derivatives up to some order k. ω in (1) is a certain real number.

It is known that the solutions of the problems (1), (2), (3); (1), (2), (4); (1), (2), (5) exist and are unique.

Let λ be some complex number. Denote by $w(x, \lambda)$ the solution in the domain D of the Helmholtz equation

$$\Delta w - \lambda^2 w = f(x), \tag{6}$$

satisfying one of the conditions

$$w\big|_S = 0, \tag{7}$$

$$\frac{\partial w}{\partial n}\bigg|_S = 0, \tag{8}$$

$$\left(\frac{\partial w}{\partial n} + \sigma(x) w\right)\bigg|_S = 0 \tag{9}$$

on the boundary S, and the Sommerfeld radiation condition (^{1,2}) at infinity

$$\frac{\partial w}{\partial r} + \lambda w = o\left(\frac{e^{-|\operatorname{Re}\lambda| r}}{r}\right). \tag{10}$$

Lemma 1. For every complex $\lambda \neq 0$ there exists a unique solution $w(x, \lambda)$ of any of the problems (6), (7), (10); (6), (8), (10); (6), (9), (10).

For the first and second boundary-value problems the proof of this lemma is given in ($^{1-3}$). As for the third boundary-value problem, the proof of uniqueness of the solution is carried out by a method analogous to that in (^{1}), while the proof of existence will be outlined below.

The aim of this note is to prove the following theorem:

Theorem (Principle of limiting amplitude). There exist $\alpha > 0$ and $C(\alpha) > 0$ such that

$$\left|u(x, t) - v(x) e^{i\omega t}\right| \leq C(\alpha) e^{-\alpha t} \tag{11}$$

for $t \geqslant 0$, where $u(x, t)$ is the solution of one of the exterior boundary-value problems for the wave equation (1), (2), (3); (1), (2), (4), or (1), (2), (5). The function $v(x)$, called the limiting amplitude, is the solution, respectively, of the problems (6), (7), (10); (6), (8), (10), or (6), (9), (10) for $\lambda = i\omega$.

For the first boundary-value problem (1), (2), (3) this theorem was proved by C. Morawetz, P. Lax, and R. Phillips in (^{4,5}). The limiting-amplitude principle for the first boundary-value problem was also studied in spaces of arbitrary dimension in the papers (^{6,7}); in these papers, under the same restrictions on the boundary S of the domain as in (^{4,5}), inequality (11) is proved with the factor $e^{-\alpha t}$ replaced by $1/\sqrt{t}$. A number of papers have also been devoted to the justification of the limiting-amplitude principle for the Cauchy problem, i.e., for the case when there is no reflecting body: in paper (^{8}) this problem was first posed and solved for the wave equation; in papers (^{9,10}) the question is studied of the asymptotic behavior as $t \to \infty$ of the solution of the Cauchy problem for equation (1), in which, in place of the operator Δ, there stands a linear homogeneous elliptic operator $L(\partial/\partial x)$ with constant coefficients of order $2m$. Under certain restrictions on $L(\partial/\partial x)$ (its sign is assumed chosen so that $L(i\xi) < 0$ for real ξ, $|\xi| = 1$) it is proved that $\lim_{t\to\infty} u(x,t)e^{-i\omega t} = v(x)$, where $v(x)$ is the solution of the corresponding elliptic equation $L(\partial/\partial x)v + \omega^2 v = f(x)$. Here $v(x)$ turns out to be the same solution of this equation as the solution singled out by means of the so-called "limiting absorption" principle. For the equation $\Delta u - u_{tt} + a(x)u = f(x)e^{i\omega t}$ with smooth and finite $a(x)$, the question of the limiting amplitude for the Cauchy problem was studied in (^{11}).

In proving the theorem we shall, for definiteness, dwell on the boundary condition (5) of the third boundary-value problem. Applying the Laplace transform with respect to t to equation (1) and condition (5), and using the initial conditions (2), we obtain

$$\Delta\tilde{u}(x,\lambda) - \lambda^2\tilde{u}(x,\lambda) = \frac{f(x)}{\lambda - i\omega}, \tag{12}$$

$$\left(\frac{\partial\tilde{u}(x,\lambda)}{\partial n} + \sigma(x)\tilde{u}(x,\lambda)\right)_S = 0, \tag{13}$$

where

$$\tilde{u}(x,\lambda) = \int_0^\infty u(x,t)e^{-\lambda t}\,dt,$$

and $\operatorname{Re}\lambda > 0$. From the results of I. N. Vekua (^{1}) it follows that $\tilde{u}(x,\lambda)$ satisfies the Sommerfeld radiation condition (10). From Lemma 1 it follows that

$$\tilde{u}(x,\lambda) = \frac{w(x,\lambda)}{\lambda - i\omega}, \tag{14}$$

where $w(x,\lambda)$ is the solution of problem (6), (9), (10).

Lemma 2. The function $w(x,\lambda)$ is a meromorphic function of λ, analytic for $\operatorname{Re}\lambda > -2\alpha$.

Lemma 3. There exists a number $C(\alpha) > 0$ such that, for $|\operatorname{Re}\lambda| \leqslant \alpha$, $|\operatorname{Im}\lambda| \geqslant 1$,

$$|w(x,\lambda)| \leqslant \frac{C(\alpha)}{|\lambda|}. \tag{15}$$

With the help of these lemmas the proof of the theorem is completed as follows. The function $u(x,t)$ is recovered from the function $\tilde{u}(x,\lambda)$ by the Mellin formula

$$u(x,t) = \frac{1}{2\pi i}\int_L \tilde{u}(x,\lambda)e^{\lambda t}\,d\lambda,$$

where L is the straight line $\operatorname{Re}\lambda = a$. By virtue of (14) and Lemmas 2 and 3 we have

$$u(x,t) = w(x,i\omega)e^{i\omega t} + \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=-\alpha} e^{\lambda t}\frac{w(x,\lambda)}{\lambda - i\omega}\,d\lambda = v(x)e^{i\omega t} + z(x,t),$$

since $w(x,i\omega) = v(x)$. By virtue of Lemma 3,

$$|z(x,t)| \leq C(\alpha)e^{-\alpha t}\int_{\operatorname{Re}\lambda=-\alpha}\frac{|d\lambda|}{|\lambda|\,|\lambda - i\omega|} = C_1(\alpha)e^{-\alpha t},$$

which is what had to be established.

Let us dwell briefly on the proofs of the lemmas.

With the aid of Green' s formula and the radiation condition (10), we have

$$w(x,\lambda) = \iiint_{D\cap\Omega} f(y)U(x-y,\lambda)\,dy- \tag{16}$$

$$-\iint_S w(y,\lambda)\left[\sigma(y)U(x-y,\lambda) + \frac{\partial U(x-y,\lambda)}{\partial n_y}\right]dy = F(x,\lambda) - \theta(x,\lambda),$$

where

$$U(x-y,\lambda) = \frac{e^{-\lambda|x-y|}}{4\pi|x-y|}$$

is the fundamental solution of the Helmholtz equation.

Passing in (16) to the limit as $x \to X \in S$, we obtain, by the properties of the double-layer potential, the Fredholm integral equation

$$w(X,\lambda) = 2F(X,\lambda) - 2\iint_S w(y,\lambda)\left[\sigma(y)U(X-y,\lambda) + \frac{\partial U(X-y,\lambda)}{\partial n_y}\right] dy$$

for the unknown function $w(X,\lambda)$. It can be shown that this equation is always solvable.

Below we shall need the following property of the function $F(x,\lambda)$.

Lemma 4. If $\nabla^r f(x)\big|_S = 0,\ r = 0,\dots,k$, then for every $\mu > 0$ there is a $C_1(\mu) > 0$ such that the entire function $F(x,\lambda)$ of λ (see (16)) in the region $|\operatorname{Im}\lambda| \geq 1,\ |\operatorname{Re}\lambda| \leq \mu$ satisfies the inequality

$$|F(x,\lambda)| + |\nabla_x F(x,\lambda)| \leq C_1(\mu)/|\lambda|^k.$$

Using Lemma 4, from (16) we obtain that the function $\theta(x,\lambda)$, being a solution of the homogeneous Helmholtz equation (6), satisfies condition (10) and the boundary condition

$$(\partial\theta/\partial n + \sigma(x)\theta)\big|_S = \varphi(x,\lambda)\big|_S, \tag{17}$$

where $\varphi(x,\lambda)$ is a sufficiently smooth function in x and an entire function in λ, satisfying, for $|\operatorname{Im}\lambda| \geq 1,\ |\operatorname{Re}\lambda| \leq \alpha$, the inequality

$$|\varphi(x,\lambda)| \leq C_2(\alpha)/|\lambda|^k. \tag{18}$$

For the function $\theta(x,\lambda)$ the following basic lemma is valid.

Lemma 5. If $\theta(x,\lambda)$ is a solution of equation (6) with $f \equiv 0$, satisfies the boundary condition (17) and condition (10), then for $p > 3$ the inequality

$$\|\theta(x,\lambda)\|_{L_p(D)} \leq C_3(p)\|\varphi(x,\lambda)\|_{L_p(S)} \tag{19}$$

holds, with a constant $C_3(p)$ independent of λ.

Taking into account that in the domain D, $\lambda^2\theta = \Delta\theta$, and also the known embedding theorems and (16), we obtain

$$\|w(x,\lambda)\|_{L_p(S)} \leq C_4|\lambda|^2\|\varphi(x,\lambda)\|_{L_p(S)}.$$

If now in Lemma 4 and inequality (18) we put $k = 4$ and use equality (16) once again, we obtain estimate (15) of Lemma 3. Since the function $w(x,\lambda)$, $x \in S$, is meromorphic in λ, as follows from Fredholm theory, the function $w(x,\lambda)$ for

$x \in D$ will also be meromorphic in λ. Therefore Lemma 2 follows from Lemma 3.

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References

1. I. N. Vekua, *Tr. Tbilissk. matem. inst.*, **12**, 105 (1943).
2. V. D. Kupradze, Boundary problems of the theory of oscillations and integral equations, M., 1950.
3. F. Rellich, *Jahresber. Deutsch. Math. Ver.*, **53**, 1, H. 1, 57 (1943).
4. C. S. Morawetz, *Comm. Pure and Appl. Math.*, **14**, 561 (1961).
5. P. Lax, C. Morawetz, R. Phillips, *Comm. Pure and Appl. Math.*, **16**, No. 4, 477 (1963).
6. R. Buchal, *J. Math. and Mech.*, **12**, No. 2, 225 (1963).
7. E. Zachmanoglou, *Arch. Rat. Mech. and Analysis*, **14**, No. 4, 312 (1963).
8. A. N. Tikhonov, A. A. Samarskii, *ZhETF*, **18**, No. 2, 243 (1948).
9. G. I. Basse, A. G. Kostyuchenko, *Vestn. Mosk. univ., ser. matem.*, No. 5, 153 (1959).
10. M. M. Gekhtman, *DAN*, **153**, No. 1, 20 (1963).
11. O. A. Ladyzhenskaya, *UMN*, **12**, issue 3, 161 (1957).

Note: Figure translations are in progress. See original paper for figures.

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