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Abstract

Full Text

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DUAL BANACH SPACES CONTAINING SUBSPACES OF CHARACTERISTIC ZERO

(Presented by Academician A. I. Mal' tsev on 2 IX 1963)

Let us consider a Banach space E and its dual space E' . Denote by M' a vector subspace of the space E' , everywhere dense in the weak topology $\sigma(E', E)$, and by $\|x\|^*$ the norm of an element $x \in E$ generated by the functionals from the subspace M' :

$$\|x\|^* = \sup_{x' \neq 0, x' \in M'} \frac{|\langle x, x' \rangle|}{\|x'\|}. \quad (1)$$

The subspace M' is called a **subspace with zero characteristic** if the norm $\|x\|^*$ is not equivalent to the original norm $\|x\|$, defined in the space E (see (2)).

In (5) it was shown that, in the case of a reflexive space E , the norm (1) always coincides with the original norm $\|x\|$, so that the question of the existence of subspaces M' with zero characteristic is naturally posed only for nonreflexive Banach spaces.

Theorem 1. *If a Banach space E is isomorphic to the Cartesian product $E \times F$, where F is some nonreflexive Banach space, then the dual space E' contains a subspace M' , everywhere dense in the weak topology $\sigma(E', E)$, whose characteristic is zero.*

Lemma 1. *Let E be a Banach space containing a separable subspace E_1 such that in the dual space E'_1 there exists a subspace of characteristic zero, everywhere dense in the topology $\sigma(E'_1, E_1)$. Then the dual space E' also contains a subspace of characteristic zero, everywhere dense in the topology $\sigma(E', E)$.*

Lemma 2. *The lower bound of two separate locally convex topologies \mathfrak{T} and \mathfrak{T}_1 , given on the vector space E , in the set of all topologies compatible with the structure of the vector space E , is a locally convex topology.*

Proof. Let T_1 be the lower bound of the topologies \mathfrak{T} and \mathfrak{T}_1 in the set of all locally convex topologies, and let T be the lower bound of \mathfrak{T} and \mathfrak{T}_1 in the set of all topologies compatible with the structure of the vector space E .

Denote by \mathfrak{B} a fundamental system of neighborhoods of zero of the topology T_1 , whose existence is asserted in (1) (see Proposition 5, p. 27). Consider an arbitrary neighborhood $V \in \mathfrak{B}$, and let W be a neighborhood of zero from \mathfrak{B} such that $W + W \subset V$. The neighborhood W contains convex neighborhoods

of zero \mathcal{U} and \mathcal{U}_1 in the topologies \mathfrak{T} and \mathfrak{T}_1 , respectively, since the topology T_1 is weaker than the topologies \mathfrak{T} and \mathfrak{T}_1 . It is easy to see that the convex hull K of the set $\mathcal{U} \cup \mathcal{U}_1 \subset W$ belongs to the set $W + W$. Indeed, if $z \in K$, then $z = tx + (1-t)x_1$ ($0 \leq t \leq 1$), where $x \in \mathcal{U}$, $x_1 \in \mathcal{U}_1$; but $tx \in \mathcal{U} \subset W$, $(1-t)x_1 \in \mathcal{U}_1 \subset W$, since \mathcal{U} and \mathcal{U}_1 are convex sets, so that $tx + (1-t)x_1 \in W + W$, and the inclusion $K \subset W + W$ is proved.

It remains to note that the sets K form a fundamental system of neighborhoods of zero of the topology T , when \mathcal{U} and \mathcal{U}_1 run through fundamental systems of neighborhoods of zero in the topologies \mathfrak{T} and \mathfrak{T}_1 . Lemma 2 is proved.

Proof of Theorem 1. It follows from the hypotheses of the theorem that in the Banach space E one can select a sequence of nonreflexive closed subspaces $F_1, F_2, \dots, F_n, \dots$, possessing the following properties:

1. $F_i \cap F_j = \theta$, if $i \neq j$.
2. For every natural n the direct sum $E_n = F_1 + F_2 + \dots + F_n$ is closed in the space E .

Without loss of generality we may assume that each of the subspaces $F_1, F_2, \dots, F_n, \dots$ is separable. From the results of paper ⁽⁵⁾ it follows that on the subspace F_n one can define a new normed topology $T_{(n)}$, weaker than the original topology of the space F_n , in such a way that the closure of the unit ball $S_1^{(n)}$ of the space F_n in the topology $T_{(n)}$ contains elements x whose norms $\|x\|$ exceed the number n .

Consider on the subspace $E_n = F_1 + F_2 + \dots + F_n$ the direct sum P_n of the topologies $T_{(1)}, T_{(2)}, \dots, T_{(n)}$, and denote by \tilde{E} the union of all subspaces E_n . It is easy to see that, for every n , the topology induced on E_n by the topology P_{n+1} coincides with the topology P_n . We may therefore define on the vector space \tilde{E} the strict inductive limit \mathcal{T} of the normed topologies P_n .

We shall further denote by \mathcal{T}_1 the topology induced on \tilde{E} by the original topology of the space E , and by T_1 the lower bound of the topologies \mathcal{T} and \mathcal{T}_1 in the set of all locally convex topologies. It is not hard to note that T_1 is a separated locally convex topology on the space \tilde{E} .

If \mathcal{U} and \mathcal{U}_1 are two neighborhoods from the fundamental system of neighborhoods of zero in the topologies \mathcal{T} and \mathcal{T}_1 , then the convex hull K of the set $\mathcal{U} \cup \mathcal{U}_1$ will form a fundamental system of neighborhoods of zero of the topology T_1 , as \mathcal{U} and \mathcal{U}_1 range over the set of all neighborhoods from the fundamental system of neighborhoods of zero of the topologies \mathcal{T} and \mathcal{T}_1 .

The topology T_1 is the upper bound for the set of all locally convex topologies that are weaker than the topology \mathcal{T}_1 of the space \tilde{E} and induce on each of the subspaces E_n the topology P_n .

The closure of the unit ball $\tilde{S} = \tilde{E} \cap S$, where S is the unit ball of the space E , in the topology T_1 is an unbounded set in the norm of the space E . If we prove that

the completion of the vector space \widetilde{E} , endowed with the topology T_1 , contains the closure \widetilde{E}_1 of the subspace \widetilde{E} in the Banach space E , then it will follow that in the conjugate space \widetilde{E}'_1 there exists an everywhere dense, in the topology $\sigma(\widetilde{E}'_1, \widetilde{E}_1)$, subspace of characteristic zero. For this purpose consider the lower bound T of the topologies \mathcal{T} and \mathcal{T}_1 in the set of all topologies compatible with the structure of the vector space \widetilde{E} , and show that the completion \widetilde{E} , endowed with the topology T , contains \widetilde{E}_1 .

The latter condition is equivalent to the following assertion: every sequence (x_n) fundamental in the norm \mathcal{T}_1 and converging to zero in the topology T tends to zero in the topology \mathcal{T}_1 . It is easy to see that every net (z_n) converging to zero in the topology T has the form $z_n = x_n + y_n$, where x_n and y_n are nets converging to zero in the topologies \mathcal{T} and \mathcal{T}_1 , respectively. If the sequence (x_n) converges to zero in the topology \mathcal{T} , then it belongs to one of the subspaces E_n . To prove this, note that each of the subspaces E_n is closed in E_{n+1} (in the topology P_{n+1}); therefore E_n is also closed in the topology \mathcal{T} (see ⁽¹⁾, Remark 2, p. 88). The sequence (x_n) is a bounded set in the topology \mathcal{T} , since (x_n) is a fundamental sequence. By the known theorem on bounded sets in a strict inductive limit (see ⁽¹⁾, Proposition 6, p. 151), we may assert that the sequence (x_n) is contained in one of the subspaces E_n .

The space E_n , endowed with the induced topology \mathcal{T}_1 , is ...

Banach space; therefore the sequence (x_n) converges to zero in the norm \mathcal{T}_1 . Hence it follows that the sequence $z_n = x_n + y_n$ converges to zero in the topology \mathcal{T}_1 , so that the completion of the space \widetilde{E} , endowed with the topology T , contains the Banach space \widetilde{E}_1 .

From Lemma 2 it follows that the topologies T and T_1 are equivalent. Thus, on the space \widetilde{E}_1 there is defined a separable locally convex topology T_1 , weaker than the original topology of the space \widetilde{E}_1 , and the closure of the unit ball \widetilde{S}_1 of the space \widetilde{E}_1 in the topology T_1 is unbounded in the norm of \widetilde{E}_1 . Consequently, \widetilde{E}_1 has an everywhere dense, in the weak topology, subspace of characteristic zero, and therefore (Lemma 1) the conjugate space E' contains an analogous subspace. The theorem is proved.

Corollary 1. *If a nonreflexive Banach space E is isomorphic to its Cartesian square $E \times E$, then the conjugate space E' contains an everywhere dense, in the topology $\sigma(E', E)$, subspace of characteristic zero.*

With the aid of Theorem 1, Lemma 1, and Theorem 2 from the paper of R. James ⁽³⁾, it is not difficult to prove

Corollary 2. *Let E be a Banach space that contains a nonreflexive subspace E_1 with an unconditional basis. Then in the conjugate space E' there exists an everywhere dense, in the topology $\sigma(E', E)$, subspace M' whose characteristic is equal to zero.*

The results presented make it possible to establish the existence of subspaces with zero characteristic, everywhere dense in the topology $\sigma(E', E)$, in the conjugate spaces of the Banach spaces c_0 , c , l_1 , m , $C(0, 1)$, $L_1(0, 1)$, $M(0, 1)$, C_α (Hölder space), etc.

A Banach space E is called **quasi-reflexive** if the quotient space E''/E is finite-dimensional ⁽⁴⁾. It is easy to see that, for a quasi-reflexive Banach space, the conditions of Theorem 1 are not satisfied.

Theorem 2. *Every subspace M' of the conjugate space E' of a quasi-reflexive Banach space, everywhere dense in the weak topology $\sigma(E', E)$, has characteristic $r_{M'}$ different from zero.*

Definition 1. We shall say that a Banach space E has the R -property if it contains an infinite-dimensional reflexive subspace.

Theorem 3 (*criterion of quasi-reflexivity of a separable Banach space*). *Let E be a separable Banach space. In order that E be quasi-reflexive, it is necessary and sufficient that the conjugate space have the R -property and that the second conjugate space E'' contain no subspaces with zero characteristic, everywhere dense in the topology $\sigma(E'', E')$.*

The question of the existence of subspaces $M' \subset E'$, everywhere dense in the weak topology $\sigma(E', E)$ and with zero characteristic in an arbitrary non-quasi-reflexive Banach space, remains open.

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Note: Figure translations are in progress. See original paper for figures.

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