MECHANICS

1964

SovietRxiv

Abstract

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MECHANICS

M. V. Rein

ON ONE GENERAL METHOD FOR REDUCING THE ORDER OF A HAMILTONIAN SYSTEM WITH A KNOWN INTEGRAL

(Presented by Academician A. A. Dorodnitsyn on 14 I 1964)

Let a dynamical system be defined by the Hamiltonian function $\mathcal{H}(p_i, q_i, t)$ and have the integral

$$f(q_i, p_i, t) = \text{const} \quad (i = 1, \dots, n). \tag{1}$$

We shall seek a canonical transformation with generating function $W(q_i, \tilde{q}_j, \lambda, t)$:

$$\tilde{p}_0 = -\frac{\partial W}{\partial \lambda}, \qquad \tilde{p}_j = -\frac{\partial W}{\partial \tilde{q}_j}, \qquad p_i = \frac{\partial W}{\partial q_i}, \qquad \widetilde{\mathcal{H}} = \mathcal{H} + \frac{\partial W}{\partial t} \tag{2}$$

$$(i = 1, \dots, n;\ j = 1, \dots, n-1);$$

$$\begin{vmatrix} \dfrac{\partial^2 W}{\partial q_1 \partial \lambda} & \cdots & \dfrac{\partial^2 W}{\partial q_n \partial \lambda} \\ \dfrac{\partial^2 W}{\partial q_1 \partial \tilde{q}_1} & \cdots & \dfrac{\partial^2 W}{\partial q_n \partial \tilde{q}_1} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial^2 W}{\partial q_1 \partial \tilde{q}_{n-1}} & \cdots & \dfrac{\partial^2 W}{\partial q_n \partial \tilde{q}_{n-1}} \end{vmatrix} \neq 0, \tag{3}$$

which transforms the function $f(q_i, p_i, t)$ into the momentum $\tilde{p}_0$ corresponding to the coordinate λ. Then the relation $f(q_i, p_i, t) = \tilde{p}_0$, taking (2) into account, leads to the partial differential equation for W:

$$f\left(q_i, \frac{\partial W}{\partial q_i}, t\right) = -\frac{\partial W}{\partial \lambda} \qquad (i = 1, \dots, n). \tag{4}$$

It is easy to see that (4) is a complete analogue of the Hamilton–Jacobi equation, where the role of the Hamiltonian function is played by the function $f(q_i, p_i, t)$, and the role of time by λ, while t is a parameter. The problem reduces to finding, for equation (4), an incomplete integral $W(q_i, \tilde{q}_j, \lambda, t)$ $(i = 1, \dots, n;\ j = 1, \dots, n-1)$, depending on $n-1$ arbitrary constants $\tilde{q}_j$ and satisfying condition (3). We shall show that such an integral exists. Indeed, the general solution of the Hamilton equations corresponding to (4),

$$\frac{dq_i}{d\lambda} = \frac{\partial f}{\partial p_i}, \qquad \frac{dp_i}{d\lambda} = -\frac{\partial f}{\partial q_i} \qquad (i = 1, \dots, n) \tag{5}$$

makes it possible to find, for equation (4), a complete integral $W(q_i, \tilde{q}_i, \lambda, t)$ $(i = 1, \dots, n)$ (an integral depending on n arbitrary constants $\tilde{q}_i$, with $\det\left(\frac{\partial^2 W}{\partial q_i \partial \tilde{q}_k}\right)_{i,k=1}^{n} \neq 0$), for which not all $\partial^2 W/\partial q_i \partial \lambda$ are identically equal to zero. (The latter condition is satisfied, for example, by the principal Hamilton function for system (5).)

Consider the rectangular matrix

$$\left\| \begin{matrix} \dfrac{\partial^2 W}{\partial q_1 \partial \lambda} & \cdots & \dfrac{\partial^2 W}{\partial q_n \partial \lambda} \\ \dfrac{\partial^2 W}{\partial q_1 \partial \tilde{q}_1} & \cdots & \dfrac{\partial^2 W}{\partial q_n \partial \tilde{q}_1} \\ \cdot & \cdot & \cdot \\ \dfrac{\partial^2 W}{\partial q_1 \partial \tilde{q}_n} & \cdots & \dfrac{\partial^2 W}{\partial q_n \partial \tilde{q}_n} \end{matrix} \right\|. \tag{5a}$$

Since

$$\det\left(\frac{\partial^2 W}{\partial q_i \partial \tilde{q}_k}\right)_{i,k=1}^{n} \neq 0$$

and not all $\partial^2 W/\partial q_i \partial \lambda$ are zero, the first row of (5a) will be a linear combination of the remaining rows, the coefficients α_i of which are not all zero; moreover, one may assume (in general, by renumbering the $\tilde{q}_k$) that the coefficient α_n at the last row of (5a) is nonzero. Then, deleting the last row of (5a), we obtain a matrix with determinant (3) equal to

$$\alpha_n \det\left(\frac{\partial^2 W}{\partial q_i \partial \tilde{q}_k}\right)_{i,k=1}^{n} \neq 0.$$

Let us now consider an integral of a special form—linear with respect to the coordinates,

$$f = \sum_{i=1}^{n} f_i(p_k, t) q_i + g(p_k, t) \quad (k = 1, \dots, n). \tag{6}$$

Then equation (4) will have the form

$$\sum_{i=1}^{n} f_i \left(\frac{\partial W}{\partial q_k}, t \right) q_i + g \left(\frac{\partial W}{\partial \tilde{q}_k}, t \right) = -\frac{\partial W}{\partial \lambda}. \tag{7}$$

In this case it suffices to consider, from system (5), only the equations for the momenta

$$\frac{dp_i}{d\lambda} = -f_i(p_k, t) \quad (i, k = 1, \dots, n), \tag{8}$$

which form a closed system.

A solution of equation (7) satisfying (3) is obtained in the form

$$W = \sum_{i=1}^{n} p_i(t, \lambda, \tilde{q}_j) q_i - \int g(p_i(t, \lambda, \tilde{q}_j), t) \, d\lambda$$

$$(i = 1, \dots, n; \ j = 1, \dots, n-1), \tag{9}$$

where $p_i(t, \lambda, \tilde{q}_j)$ is a solution of system (8) depending on $n-1$ arbitrary constants $\tilde{q}_j$.

The case of an integral linear with respect to the momenta,

$$f = \sum_{i=1}^{n} f_i(q_k, t) p_i + g(q_k, t) \quad (k = 1, \dots, n) \tag{6'}$$

is reduced to the one just considered after the transformation of coordinates into momenta and leads to the need to solve the system of equations

$$\frac{dq_i}{d\lambda} = f_i(q_k, t) \quad (i, k = 1, \dots, n). \tag{8'}$$

The transformation obtained here for the integrals (6), ($6'$) is a generalization of the transformation for a linear and homogeneous integral with respect to the momenta ($g(q_i, t) \equiv 0$), given in ([1]) and based on the solution of the system

$$dq_1/f_1 = dq_2/f_2 = \dots = dq_n/f_n$$

(which is essentially equivalent to system (8′)), to nonhomogeneous integrals ($g \neq 0$).

Let us apply the method described here for lowering the order of a Hamiltonian system to the three-body problem. We shall assume that, on the basis of the integrals of the center of gravity, the system of equations has been reduced by one of the known methods to a system of order 12 with Hamiltonian function $\mathcal{H}(q_i, p_i) = \text{const}$ ($i = 1, \dots, 6$), having 3 integrals of the moments of quantity of motion:

$$f_1 = q_2p_1 - q_1p_2 + q_5p_4 - q_4p_5 = \text{const}, \quad (10)$$
$$f_2 = q_3p_2 - q_2p_3 + q_6p_5 - q_5p_6 = \text{const}, \quad (10')$$
$$f_3 = q_1p_3 - q_3p_1 + q_4p_6 - q_6p_4 = \text{const}, \quad (10'')$$

which are integrals of type (6). We shall seek a transformation (2) which converts the integral (10) into the impulse p_0. This leads to the necessity of solving the system (8), which in the present case has the form

$$\frac{dp_1}{d\lambda} = p_2, \qquad \frac{dp_2}{d\lambda} = -p_1, \qquad \frac{dp_4}{d\lambda} = p_5, \qquad \frac{dp_5}{d\lambda} = -p_4, \qquad \frac{dp_6}{d\lambda} = \frac{dp_3}{d\lambda} = 0. \tag{11}$$

The general solution of (11) is

$$\begin{aligned} p_1 &= (\tilde{q}_1 \sin\lambda + \tilde{q}_6 \cos\lambda), & p_2 &= (\tilde{q}_1 \cos\lambda - \tilde{q}_6 \sin\lambda), \\ p_4 &= (\tilde{q}_2 \sin\lambda + \tilde{q}_3 \cos\lambda), & p_5 &= (\tilde{q}_2 \cos\lambda - \tilde{q}_3 \sin\lambda), \\ p_3 &= \tilde{q}_4, & p_6 &= \tilde{q}_5. \end{aligned} \tag{12}$$

Therefore the complete integral of equation (7) may be taken in the form

$$W = (\tilde{q}_1 \sin\lambda + \tilde{q}_6 \cos\lambda)q_1 + (\tilde{q}_1 \cos\lambda - \tilde{q}_6 \sin\lambda)q_2 +$$

$$+ (\tilde{q}_2 \sin\lambda + \tilde{q}_3 \cos\lambda)q_4 + (\tilde{q}_2 \cos\lambda - \tilde{q}_3 \sin\lambda)q_5 + \tilde{q}_4 q_3 + \tilde{q}_5 q_6. \tag{13}$$

The required generating function can be obtained in various ways: impose on the 6 arbitrary constants a relation $F(\tilde{q}_1, \dots, \tilde{q}_6) = 0$ and, with its aid, eliminate from (13) one of the arbitrary constants so that condition (3) is satisfied; introduce new arbitrary constants c_i in (13) by the replacement $\tilde{q}_j = \tilde{q}_j(c_i)$ and eliminate one of them so that, with respect to the remaining 5, condition (3) is satisfied; or simply fix one of the constants $\tilde{q}_j$ in (13), as we shall do, putting $\tilde{q}_6 = 0$. In this case it is easy to verify that condition (3) is satisfied* and the generating function has the form

$$W = q_1\tilde{q}_1 \sin\lambda + q_2\tilde{q}_1 \cos\lambda + q_4(\tilde{q}_2 \sin\lambda + \tilde{q}_3 \cos\lambda) +$$

$$+ q_5(\tilde{q}_2 \cos\lambda - \tilde{q}_3 \sin\lambda) + q_3\tilde{q}_4 + q_6\tilde{q}_5. \qquad (14)$$

The transformation with generating function (14) brings any of the considered systems of order 12 to a system with $\widetilde{\mathcal{H}}(\tilde{p}_j, \tilde{q}_j, \tilde{p}_0) = \mathcal{H} = \text{const}$ $(j = 1, \dots, 5)$, having the cyclic coordinate λ and the integrals

$$f_2 = (\tilde{q}_4\tilde{p}_1 - \tilde{p}_4\tilde{q}_1 + \tilde{p}_2\tilde{q}_5 - \tilde{q}_2\tilde{p}_5)\cos\lambda +$$

$$+ \left(\tilde{p}_5\tilde{q}_3 - \tilde{q}_5\tilde{p}_3 - \frac{\tilde{p}_0 + \tilde{p}_2\tilde{q}_3 - \tilde{p}_3\tilde{q}_2}{\tilde{q}_1}\,\tilde{q}_4\right)\sin\lambda = \text{const},$$

$$f_3 = -(\tilde{q}_4\tilde{p}_1 - \tilde{p}_4\tilde{q}_1 + \tilde{p}_2\tilde{q}_5 - \tilde{q}_2\tilde{p}_5)\sin\lambda +$$

$$+ \left(\tilde{p}_5\tilde{q}_3 - \tilde{q}_5\tilde{p}_3 - \frac{\tilde{p}_0 + \tilde{p}_2\tilde{q}_3 - \tilde{p}_3\tilde{q}_2}{\tilde{q}_1}\,\tilde{q}_4\right)\cos\lambda = \text{const}, \qquad (15)$$

which contain the cyclic coordinate. Eliminating λ from the integrals (15), we obtain an integral independent of the cyclic coordinate:

$$(\tilde{q}_4\tilde{p}_1 - \tilde{p}_4\tilde{q}_1 + \tilde{p}_2\tilde{q}_5 - \tilde{q}_2\tilde{p}_5)^2 +$$

$$+ \left(\tilde{p}_5\tilde{q}_3 - \tilde{q}_5\tilde{p}_3 - \frac{\tilde{p}_0 + \tilde{p}_2\tilde{q}_3 - \tilde{p}_3\tilde{q}_2}{\tilde{q}_1}\,\tilde{q}_4\right)^2 = \text{const}, \qquad (16)$$

* We note that the constants $\tilde{q}_4$ or $\tilde{q}_5$ cannot be fixed, since condition (3) will not be fulfilled. (When any of the constants $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_6$ is fixed, condition (3) is fulfilled.)

on the basis of which the order can be reduced to 8.

However, one may always assume $f_2 = f_3 = 0$. Then two conditions must be satisfied simultaneously

$$\begin{aligned} f_1' &= \tilde{q}_4\tilde{p}_1 - \tilde{p}_4\tilde{q}_1 + \tilde{p}_2\tilde{q}_5 - \tilde{q}_2\tilde{p}_5 = 0, \\ f_2' &= \tilde{q}_3\tilde{p}_5 - \tilde{q}_5\tilde{p}_3 - \frac{\tilde{p}_0 + \tilde{p}_2\tilde{q}_3 - \tilde{p}_3\tilde{q}_2}{\tilde{q}_1}\,\tilde{q}_4 = 0. \end{aligned} \qquad (17)$$

Therefore, in order to transform the system on the basis of the integral (16), we proceed as follows: find the transformation (2) that carries the function

f_1' into the momentum $\tilde{p}_0$ corresponding to the coordinate λ_1. Then all the new coordinates and momenta will enter, in general, into the new Hamiltonian function $\widetilde{\mathcal{H}}$, while the relations (17) will become the relations $\tilde{p}_0 = 0$, $f_2' = 0$; moreover $d\tilde{p}_0/dt = \partial\widetilde{\mathcal{H}}/\partial\lambda_1 = f_2'\mathcal{H}_1 = 0$, where $\mathcal{H}_1$ is some function of the coordinates and momenta. After eliminating λ_1 from $\widetilde{\mathcal{H}} = \text{const}$ by means of the relation $f_2' = 0$, we obtain a conservative system having the cyclic coordinate λ_1. Applying the transformations considered here to a system of the 12th order with Hamiltonian function

$$\begin{aligned}\mathcal{H} = {} & \frac{1}{2\mu}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2\mu'}(p_4^2 + p_5^2 + p_6^2) - m_1 m_2 (q_1^2 + q_2^2 + q_3^2)^{-1/2} \\ & - m_1 m_3 \left\{ q_4^2 + q_5^2 + q_6^2 + \frac{2m_2}{m_1 + m_2}(q_1 q_4 + q_2 q_5 + q_3 q_6) \right. \\ & \left. + \left(\frac{m_2}{m_1 + m_2}\right)^2 (q_1^2 + q_2^2 + q_3^2) \right\}^{-1/2} \\ & - m_2 m_3 \left\{ q_4^2 + q_5^2 + q_6^2 - \frac{2m_1}{m_1 + m_2}(q_1 q_4 + q_2 q_5 + q_3 q_6) \right. \\ & \left. + \left(\frac{m_1}{m_1 + m_2}\right)^2 (q_1^2 + q_2^2 + q_3^2) \right\}^{-1/2},\end{aligned}$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$, $\mu' = \dfrac{m_3(m_1 + m_2)}{m_1 + m_2 + m_3}$, we arrive at an 8th-order system determined by the Hamiltonian function

$$\begin{aligned}\mathcal{H} = {} & \frac{1}{2\mu}\left\{ p_1^2 + \frac{(p_2 q_3 - p_3 q_2)^2 + \left(p_2 q_4 - q_2 p_4 + \sqrt{\tilde{p}_0^2 - (q_3 p_4 - p_3 q_4)^2}\right)^2}{q_1^2} \right\} \\ & + \frac{1}{2\mu'}(p_2^2 + p_3^2 + p_4^2) - \frac{m_1 m_2}{q_1} \\ & - m_1 m_3 \left\{ q_2^2 + q_3^2 + q_4^2 + \frac{2m_2 q_1 q_2}{m_1 + m_2} + \left(\frac{m_2 q_1}{m_1 + m_2}\right)^2 \right\}^{-1/2} \\ & - m_2 m_3 \left\{ q_2^2 + q_3^2 + q_4^2 - \frac{2m_1 q_1 q_2}{m_1 + m_2} + \left(\frac{m_1 q_1}{m_1 + m_2}\right)^2 \right\}^{-1/2},\end{aligned}$$

whose order can be reduced by another 2 owing to its conservativity.

In the literature there is no general rule for reducing the order of a Hamiltonian system by 2 from a known integral. Thus, in the three-body problem, the reduction of the order of the equations of motion by 2 from one known integral is based on physical (geometrical) meaning. In the present paper the author has attempted to fill this gap, employing for this purpose the well-developed apparatus of analytical mechanics.

In conclusion, I take the opportunity to express my sincere gratitude to Corresponding Member of the Academy of Sciences of the USSR V. V. Struminskii for his attention to the work.

Received
8 I 1964

CITED LITERATURE

1. E. T. Whittaker, *Analytical Dynamics*, 1937.

Note: Figure translations are in progress. See original paper for figures.