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Abstract

Full Text

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ON THE APPROXIMATION OF SINGULAR SOLUTIONS OF THE CHAPLYGIN EQUATION

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As follows from the results of the note ([1]), if for the Chaplygin equation

$$z_{\theta\theta} - z_{\sigma\sigma} - b(\sigma) z_\sigma = 0, \qquad b(\sigma) = \left[\ln \sqrt{K(\sigma)}\right]_\sigma = \sum_{n=0}^{\infty} b_n \sigma^{2/3n-1} \tag{1}$$

one considers, in the region $\sigma \geq 0$, the singular Cauchy problem

$$z(\theta, 0) = \tau(\theta), \qquad z_\eta(\theta, 0) = v(\theta), \qquad \eta = -\left({}^3/_2\sigma\right)^{2/3}, \tag{2}$$

and seeks its solution $z(\theta, \sigma)$ in the integral form

$$z(\theta, \sigma) = \int_{\theta-\sigma}^{\theta+\sigma} G(\theta - \alpha, \sigma)\tau(\alpha)\, d\alpha + \int_{\theta-\sigma}^{\theta+\sigma} \overline{G}(\theta - \alpha, \sigma) v(\alpha)\, d\alpha, \tag{3}$$

then $G(\theta, \sigma)$ and $\overline{G}(\theta, \sigma)$ can be approximated in a neighborhood of the line $\sigma = 0$ by the series

$$G(\theta, \sigma) = \sum_{n=0}^{\infty} G_n(\theta, \sigma), \qquad \overline{G}(\theta, \sigma) = \sum_{n=0}^{\infty} \overline{G}_n(\theta, \sigma), \tag{4}$$

in which $G_0 = \overline{\gamma}_1 \sigma^{2/3} r^{-5/3}$ and $\overline{G}_0 = -\gamma_2 r^{-1/3}$ are the values of the kernels G and $\overline{G}$ for the case $b(\sigma) = 1/3\sigma$, $G_1 = -{}^3/_4 b_1(\sigma^{2/3} G_0 - \overline{\gamma}_2 r^{-1/3})$, $G_2 = c_0 \sigma^{4/3} G_0 + c_1 \gamma_2 \sigma^{2/3} r^{-1/3} + 4c_2 \overline{\gamma}_1 r^{1/3}$, $G_3 = 2D_0 \sigma^2 G_0 + 2D_1 \gamma_2 \sigma^{4/3} r^{-1/3} + 8D_2 \gamma_1 \sigma^{2/3} r^{1/3} + 2D_3 g_3(\theta, \sigma)$, and the functions $\overline{G}_n$ $(n = 1, 2, 3)$ have the form $\overline{G}_1 = -{}^3/_4 b_1 \sigma^{2/3} \overline{G}_0$, $\overline{G}_2 = c_0 \sigma^{4/3} \overline{G}_0 + {}^8/_9 A_1 c_2 g_3$, $\overline{G}_3 = 2D_0 \sigma^2 \overline{G}_0 - {}^8/_5 t^2 D_4 r^{5/3} - 4A_1 D_3 \sigma^{2/3} g_3$. Here $\overline{\gamma}_1 = 2^{2/3}\gamma_1$, $\gamma_0 = ({}^2/_3)^{2/3}\gamma_2$, $A_1 = -{}^1/_2({}^3/_2)^{1/3}$, $r = \sqrt{\sigma^2 - \theta^2}$; the constants c_n $(n = 0, 1, 2)$ and D_n $(n = 0, 1, 2, 3, 4)$ depend only on b_1, b_2, b_3, while by $g_3(\theta, \sigma)$ is denoted the difference

$$g_3 = \theta \left[I_{t^2}\left({}^7/_6, -{}^1/_2\right) - I_{t^2}\left({}^5/_6, -{}^1/_2\right)\right], \qquad t = \frac{r}{\sigma}.$$

Each of the functions G_n and $\overline{G}_n$ $(n = 0, 1, 2, ...)$ contains terms that become infinite on the characteristics $\theta \pm \sigma = 0$, and therefore, in order to improve the convergence of the series (4) near the lines $\theta \pm \sigma = 0$, it is expedient to consider another iteration method. Substituting in (1) $z = \sigma^{1/6}\chi^{-1}u$, $\chi = \sqrt[4]{K}$, we obtain ([2])

$$T[u] = u_{\theta\theta} - u_{\sigma\sigma} - \frac{1}{3\sigma}u_\sigma + c(\sigma)u = 0, \qquad (5)$$

where

$$c(\sigma) = \frac{5}{36\sigma^2} + \chi^{-1}\chi_{\sigma\sigma} = \sum_{n=-1}^{\infty} c_n \sigma^{2n/3}.$$

We shall use the procedure indicated in ([1,3]), and instead of (4) we shall seek, in the region Ω $(\sigma \geq \theta \geq 0)$, the Riemann integrals

$$W(\theta, \sigma) = \int_0^\theta G(\alpha, \sigma)\, d\alpha, \qquad \overline{W}(\theta, \sigma) = \int_0^\theta \overline{G}(\alpha, \sigma)\, d\alpha,$$

and then, differentiating them with respect to θ, compute the kernels themselves: $G = W_\theta$, $\overline{G} = \overline{W}_\theta$. The functions $W(\theta, \sigma)$ and $\overline{W}(\theta, \sigma)$, as well as $G, \overline{G}$, satisfy equation (5), and they are uniquely determined in Ω by their boundary data

$$W(0, \sigma) = \overline{W}(0, \sigma) = 0, \qquad W(\sigma, \sigma) = S(\sigma), \qquad \overline{W}(\sigma, \sigma) = \overline{S}(\sigma). \qquad (6)$$

Here $S(\sigma)$ and $\overline{S}(\sigma)$ are the solutions of the Cauchy problem (2), (5) for the values $\tau(\theta) = 1/2$, $v(\theta) = 0$ and $\tau(\theta) = 0$, $v(\theta) = 1/2$ $(-\infty < \theta < \infty)$, respectively.

or, what is the same thing, the solutions of two initial problems

$$S_{\sigma\sigma} + \frac{1}{3\sigma}S_\sigma - c(\sigma)S = 0, \quad S(0) = \overline{S}_\eta(0) = \frac{1}{2}, \quad S_\eta(0) = \overline{S}(0) = 0. \qquad (7)$$

Put $W = \sum_{n=0}^{\infty} W_n(\theta, \sigma)$ and replace (5) by the recurrence system:

$$E[W_0] = 0, \quad E[W_1] = -c_{-1}\sigma^{-2/3}W_0, \quad E[W_2] = -c_{-1}\sigma^{-2/3}W_1 - c_0 W_0, \dots, \qquad (8)$$

where $E[W] = W_{\theta\theta} - W_{\sigma\sigma} - \frac{1}{3\sigma}W_\sigma$. Equations (7) lead, near $\sigma = 0$, to the corresponding expansions for the initial functions $S(\sigma) = \sum_{n=0}^{\infty} S_n(\sigma)$,

$$\overline{S}(\sigma)=\sum_{n=0}^{\infty}\overline{S}_n(\sigma),$$

where

$$S_0=\frac{1}{2},\quad \overline{S}_0=A_1\sigma^{2/3},\quad S_n=\mu_n\sigma^{2/3(n+1)},\quad \overline{S}_n=\overline{\mu}_n\sigma^{2/3(n+2)}$$

$$(n=1,2,...),$$

with

$$\mu_1=\frac{9}{16}c_{-1},\quad \overline{\mu}_1=\frac{3}{8}A_1c_{-1},...$$

Consequently, (6) will be satisfied if we require that, for all $n=0,1,2,...,$

$$W_n(0,\sigma)=\overline{W}_n(0,\sigma)=0,\quad W_n(\sigma,\sigma)=S_n(\sigma),\quad \overline{W}_n(\sigma,\sigma)=\overline{S}_n(\sigma).\quad (9)$$

Solving successively the boundary-value problems (8), (9), we obtain

$$W_0=\frac{1}{2}I_\xi\left(\frac{1}{2},\frac{1}{6}\right),\quad \overline{W}_0=\frac{1}{2}\eta I_\xi\left(\frac{1}{2},\frac{5}{6}\right),\quad W_1=\frac{9}{16}c_{-1}\sigma^{4/3}I_\xi\left(\frac{1}{2},\frac{7}{6}\right),\dots,\quad (10)$$

where $\xi=\theta^2/\sigma^2$. Continuing such calculations, one can, step by step, find the subsequent iterations $W_n,\overline{W}_n$ $(n=2,3,...)$, which are uniquely determined by conditions (8), (9) and are likewise expressed in the form of a linear combination of incomplete Euler beta-functions

$$I_\xi\left(\frac{1}{2},\frac{1}{6}+n\right),\quad I_\xi\left(\frac{1}{2},\frac{5}{6}+n\right)$$

$(n=0,1,2,...)$. We now differentiate (10) with respect to the variable θ; then, for the terms $G_n=(W_n)_\theta$ and $\overline{G}_n=(\overline{W}_n)_\theta$ of the series (4), we obtain

$$G_0=\gamma_1\sigma^{1/3}r^{-5/3},\quad G_1=\frac{9}{2}c_{-1}\overline{\gamma}_1r^{1/3},\quad \overline{G}_0=-\gamma_2r^{-1/3},\quad \overline{G}_1=A_1c_{-1}g_3(\theta,\sigma),...\quad (11)$$

These expressions show that, in contrast to the case (1), for (5), with the same iteration method, already the second approximations $G_1(\theta,\sigma)$ and $\overline{G}_1(\theta,\sigma)$ are not only bounded, but even vanish on the characteristics $\theta\pm\sigma=0$. Improvement of the convergence of the series (4) in a neighborhood of the lines $\theta \pm \sigma = 0$ can also be achieved by means of another iterative process, in which (5) is compared with the generalized wave equation [3]

$$Q[u] = u_{\theta\theta} - u_{\sigma\sigma} - \frac{a}{\sigma}u_\sigma + c_0 u = 0 \quad (c_0 = \text{const},\ a = 1/3). \tag{12}$$

For (12), the initial problems (7) (with $c(\sigma) = c_0,\ \eta = -(\sigma/(1-a))^{1-a}$) are solved in Bessel functions

$$S(\sigma) = \frac{1}{2}J_{\beta-1/2}\left(\sigma\sqrt{c_0}\right), \quad \overline{S}(\sigma) = \frac{1}{2}\eta J_{1/2-\beta}\left(\sigma\sqrt{c_0}\right) \quad (\beta = a/2), \tag{13}$$

therefore, in order to construct the integrals $W(\theta,\sigma)$ and $\overline{W}(\theta,\sigma)$ from their boundary values (6), (13), we introduce in (12), instead of θ,σ, the variables $\xi = \theta^2/\sigma^2$ and $\zeta = -c_0\sigma^2/4$, and, moreover, put $u = \sqrt{\xi}\,v$. Then the transformed operator Q vanishes, in particular, in the case when $v(\xi,\zeta)$ satisfies simultaneously two equations:

$$\zeta v_{\zeta\zeta} - \xi v_{\xi\zeta} + \frac{1}{2}a v_\zeta + v = 0,$$

$$\xi(1-\xi)v_{\xi\xi} + \xi\zeta v_{\xi\zeta} + \frac{1}{2}[3-(5-a)\xi]v_\xi + \frac{1}{2}\zeta v_\zeta - \frac{1}{4}(2-a)v = 0.$$

One of the solutions of such a system is, as is known [4], the function

$$H_3(A,B,B+1;\xi,\zeta) = \delta\xi^{-B}\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{-n}}{(B-A+1)_n\, n!}\,I_\zeta(B,n+1-A) \tag{14}$$

for $|\xi| < 1,\ A = 1-\beta,\ B = \frac{1}{2},\ \delta = \Gamma(1-A)\Gamma(1+B)/\Gamma(B-A+1)$.

Thus,

$$W(\beta,\theta,\sigma) = \bar{\gamma}_1\frac{\theta}{\sigma}H_3\left(1-\beta,\ {}^1/_2,\ {}^3/_2;\frac{\theta^2}{\sigma^2},-\frac{c_0\sigma^2}{4}\right), \tag{15}$$

where $\bar{\gamma}_1 = 2^{1-a}\gamma_1 = \Gamma(\beta + {}^1/_2)[\sqrt{\pi}\Gamma(\beta)]^{-1}$. In an analogous way we find

$$\bar{W}(\beta,\theta,\sigma) = \bar{\gamma}_2\eta\frac{\theta}{\sigma}H_3\left(\beta,\ {}^1/_2,\ {}^3/_2;\frac{\theta^2}{\sigma^2},-\frac{c_0\sigma^2}{4}\right), \tag{16}$$

where $\bar{\gamma}_2 = (1-a)^{1-a}\gamma_2 = \Gamma({}^3/_2 - \beta)[\sqrt{\pi}\Gamma(1-\beta)]^{-1}$.

Finally, differentiating (15) and (16) with respect to θ, we arrive at the expressions

$$G = \bar{\gamma}_1 \sigma^{1-a} r^{-a} \bar{I}_{\beta-1}(r\sqrt{c_0}), \qquad \bar{G} = -\bar{\gamma}_2 r^{-a} \bar{I}_{-\beta}(r\sqrt{c_0}).$$

We shall further seek the solution of equation (5) in the form of the series $u = \sum_{n=0}^{\infty} u_n$, whose terms are determined from the recurrence sequence

$$Q[u_0] = 0, \qquad Q[u_1] = -c_{-1}\sigma^{-2/3}u_0, \qquad Q[u_2] = -c_{-1}\sigma^{-2/3}u_1 - c_1\sigma^{2/3}u_0, \ldots, \tag{17}$$

where $Q[u]$ is operator (12) for $a = {}^1/_3$. Let us first expand in an analogous series the boundary value $S(\sigma) = W(\sigma,\sigma)$ (and, correspondingly, $\bar{S}(\sigma) = \bar{W}(\sigma,\sigma)$). For this we put in (17) $u = S(\sigma)$, $u_n = S_n(\sigma)$, which gives

$$\nabla[S_0] = 0, \qquad \nabla[S_1] = c_{-1}\sigma^{-2/3}S_0, \qquad \nabla[S_2] = c_{-1}\sigma^{-2/3}S_1 + c_1\sigma^{2/3}S_0, \ldots, \tag{18}$$

where $\nabla[S] = S'' + \frac{1}{3\sigma}S' - c_0 S$. Adjoining to (18), by virtue of requirements (7), the equalities

$$S_0(0) = {}^1/_2, \qquad S_n(0) = \frac{d}{d\eta}S_{n-1}(0) = 0 \quad (n = 1, 2, \ldots),$$

we obtain

$$S_0 = {}^1/_2 \bar{I}_{-1/3}(\sigma\sqrt{c_0}), \qquad S_1 = {}^3/_2 \frac{c_{-1}}{c_0}\sigma^{1/3}S_0' = {}^9/_{16} c_{-1}\sigma^{4/3}\bar{I}_{2/3}(\sigma\sqrt{c_0}). \tag{19}$$

Analogously to this, from the subsequent equations (18) the solutions $S_2(\sigma), S_3(\sigma), \ldots$ are also uniquely determined. Now, in order to compute $W(\theta,\sigma)$ from its data (5), (6), we require that the terms of the series $W = \sum_{n=0}^{\infty} W_n$ satisfy equations (17) in the domain Ω, and on the boundaries of this domain assume the values (9), (19). Then, first of all, from (15) for $\beta = {}^1/_6$ we find $W_0 = W({}^1/_6, \theta, \sigma)$. Substituting this expression into the right-hand side of the second equation of the sequence (17), we obtain

$$W_1(\theta,\sigma) = {}^9/_2 c_{-1}\bar{\gamma}_1\sigma^{1/3}\frac{\theta}{\sigma}H_3\left(-{}^1/_6,\ {}^1/_2,\ {}^3/_2; \frac{\theta^2}{\sigma^2}, -\frac{c_0\sigma^2}{4}\right). \tag{20}$$

Continuing such computations, one can also reduce the solutions $W_2, W_3, \ldots$ of the subsequent boundary-value problems (9), (17) to combinations of known particular integrals of the Horn hypergeometric system of equations ([4]). In the same way, first from (16) for $\beta = {}^1/_6$ we obtain $\bar{W}_0(\theta, \sigma) = \bar{W}({}^1/_6, \theta, \sigma)$, and then (9) and (17) will give the further iterations $\bar{W}_1, \bar{W}_2, \ldots$. Differentiating the functions W_n and $\bar{W}_n$ obtained with respect to θ, we find

$$G_0 = \sigma^{2/3}\bar{\gamma}_1 r^{-5/3}\bar{I}_{-5/6}(r\sqrt{c_0}), \qquad G_1 = {}^9/_2 c_{-1}\bar{\gamma}_1 r^{-1/3}\bar{I}_{1/6}(r\sqrt{c_0}),$$
$$\bar{G}_0 = -\bar{\gamma}_2 r^{-1/3}\bar{I}_{-1/6}(r\sqrt{c_0}), \ldots \tag{21}$$

We now replace σ in (1) by is ($i = \sqrt{-1}$); then we obtain

$$z_{\theta\theta} + z_{ss} + \bar{b}(s)z_s = 0, \qquad \bar{b}(s) = \frac{1}{s}\left(b_0 - \sum_{n=1}^{\infty} b_{ns}^{2n/3}\right). \tag{22}$$

Let us find a solution of this equation, bounded in the half-plane $s \geq 0$ and tending for $s = 0$ to the prescribed piecewise-continuous function

$\tau(\theta)$, finite on the entire line $s = 0$:

$$z(\theta, 0) = \tau(\theta), \qquad -\infty < \theta < \infty. \tag{23}$$

It can be shown that the solution of this singular Dirichlet problem for the half-plane has the form

$$z(\theta, s) = \int_{-\infty}^{\infty} H(\theta - \alpha, s)\tau(\alpha)\, d\alpha, \tag{24}$$

where $H(\theta, s) = H_0(\theta, s)h(\theta, s)$, with $H_0 = \gamma s^{2/3} r^{-5/3}$, $\gamma = \Gamma(5/6)/\sqrt{\pi}\Gamma(1/3)$, $r = \sqrt{\theta^2 + s^2}$, corresponds to the case $\bar{b} = 1/3s$ (5), while $h(\theta, s)$ is a function bounded in the whole half-plane $s \geq 0$ with a finite limit $\lim_{\theta\to\infty} h(\theta, s)$ not identically equal to zero. We shall call the kernel of the Duamel integral here $w(\theta, s)$, the solution of the particular discontinuous Dirichlet problem (23):

$$w(\theta, 0) = \frac{1}{2}\operatorname{sign}\theta \quad (-\infty < \theta < \infty).$$

Then, for an odd function $w(\theta, s)$ of the variable θ, we have $w(0, s) = 0$, and therefore it will be constructed in the domain D ($0 \leq \theta < \infty,\ 0 \leq s < \infty$), if for (22) we solve the discontinuous Dirichlet problem for the quadrant D:

$$w(\theta, 0) = \frac{1}{2}, \qquad w(0, s) = 0 \quad (0 \leq \theta < \infty,\ 0 \leq s < \infty). \tag{25}$$

Furthermore, since $H(\theta, s) = H(-\theta, s)$, we have $w(\theta, s) = \int_0^\theta H(\xi, s)\, d\xi$, whence $w_\theta = H$, i.e., knowing the solution $w(\theta, s)$ of problem (25), one can then compute the kernel $H(\theta, s) = w_\theta$ in the domain D, and consequently, in view of the evenness of the function $H(\theta, s)$ with respect to the variable θ, also in the entire half-plane $s \geq 0$. We shall seek the solution of problem (22), (25) in the form of the series $w = \sum_{n=0}^{\infty} w_n$, whose terms satisfy the equations

$$E[w_0] = (w_0)_{\theta\theta} + (w_0)_{ss} + \frac{1}{3s}(w_0)_s = 0;$$

$$E[w_1] = -b_1 s^{-1/3}(w_0)_s; \qquad E[w_2] = -b_1 s^{-1/3}(w_1)_s - b_2 s^{1/3}(w_0)_s, \ldots$$

and the boundary conditions

$$w_0(\theta, 0) = \frac{1}{2}, \qquad w_{n+1}(\theta, 0) = w_n(0, s) = 0 \quad (n = 0, 1, 2, \ldots).$$

Then first of all we obtain

$$2w_0 = 1 - I_{s^2/r^2}\left(\frac{1}{3}, \frac{1}{2}\right) = I_{\theta^2/r^2}\left(\frac{1}{2}, \frac{1}{3}\right),$$

or, equivalently,

$$2w_0 = \gamma \sqrt[4]{27}\, F(\varphi, k),$$

where

$$\cos\varphi = \left[\sqrt{3} - 1 + (s/r)^{2/3}\right]\left[\sqrt{3} + 1 - (s/r)^{2/3}\right]^{-1},$$

and $F(\varphi, k)$ is the elliptic integral of the first kind with modulus

$$k = \frac{1}{2}\sqrt{2 + \sqrt{3}} = \sin 75^\circ.$$

The subsequent iterations here have the form:

$$w_1 = \frac{3}{4} b_1 s^{2/3} w_0,$$

$$w_2 = c_1 s^{4/3} w_0 + c_2\left[2\theta(2r)^{1/3} + (3\theta - r)(r + \theta)^{1/3} - (3\theta + r)(r - \theta)^{1/3}\right],$$

where

$$c_1 = \frac{9}{16} b_1^2, \qquad 16\sqrt[3]{2}\, c_2 = 9\gamma\left(\frac{3}{4} b_1^2 - b_2\right).$$

In the elliptic domain $s \geq 0$ in the case (5) (with $\sigma = is$), along with the first iteration method, one can also apply a second iterative process, in which the first approximations $H_0(\theta, s)$, $\bar{H}_0(\theta, s)$ to the kernels $H(\theta, s)$ and $\bar{H}(\theta, s)$ of the singular Dirichlet and Neumann problems are the integrals of equation (12) $(\sigma = is,\ b^2 = -c_0)$ (3):

$$H_0 = \gamma s^{2/3} r^{-5/3} \bar{K}_{5/6}(br), \qquad \bar{H}_0 = \gamma r^{-1/3} \bar{K}_{1/6}(br).$$

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Note: Figure translations are in progress. See original paper for figures.