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Abstract

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MATHEMATICS

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TRANSFORMATIONS OF THE DIRICHLET INTEGRAL AND SPATIAL MAPPINGS

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In the theory of plane mappings (conformal, quasiconformal, and more general ones), wide use has been made of an inequality expressing the so-called principle of length and area. This inequality, as well as a number of its modifications, is obtained as the result of simple transformations of the Dirichlet integral written for a plane mapping. The use of this inequality gives a method for studying a number of basic interior and boundary properties of plane mappings; moreover, the method makes it possible to go far beyond the class of conformal mappings. It is natural to pose the question of finding a corresponding method also in the theory of spatial mappings. Here we present the first results obtained in this direction.

1°. We begin with some general remarks on the problem of boundary correspondence under mappings.

Let A be a domain of an arbitrary compactum (X, ρ) (X is the set of elements, ρ the distance), B a domain of another compactum (Y, r), and let T be a topological mapping of the domain A onto the domain B: $B = T(A)$. In A we introduce a new (relative) distance ρ_A, and in B a distance r_B, in such a way that the relative metrics in A and B are equivalent to the metrics of the ambient spaces ρ and r, respectively.*

We complete A with respect to ρ_A and B with respect to r_B. The adjoining boundary elements of the domains A and B are classes of equivalent fundamental sequences in the metric spaces (A, ρ_A) and (B, r_B).

If our objects–the class of mappings, the class of admissible domains, and the relative distances–are introduced in a coordinated way** (and in this coordination lies the main difficulty!), then the question of boundary correspondence under the topological mapping $y = T(x)$, $x \in A$, $y \in B$, reduces to proving that, under the direct and inverse mappings T and T^{-1}, a fundamental sequence is carried into a fundamental sequence.

This question is solved automatically if one succeeds in finding functions $\varphi_1(\alpha)$ and $\varphi_2(\alpha)$ such that $\varphi_j(\alpha) \to 0$ as $\alpha \to 0$, and such that the two-sided estimate

$$\varphi_1[\rho_A(x', x'')] \leq r_B[T(x'), T(x'')] \leq \varphi_2[\rho_A(x', x'')] \tag{1}$$

holds for any points $x', x'' \in A$ sufficiently close with respect to ρ_A. Then T can be extended to a homeomorphism $T(\widetilde{A}) = \widetilde{B}$ of the domains A and B completed by boundary elements, and (1) will also be valid in the closed domain $\widetilde{A}$.

* That is, so that $\rho_A(x_n, x_0) \to 0$ if and only if $\rho(x_n, x_0) \to 0$ $(n \to \infty)$.

** Coordination in the choice of these objects must be expressed in the fact that admissible domains should be carried into one another by means of mappings of the chosen class, and boundary elements (which may be introduced by an infinite number of inequivalent methods) should be named in such a way that boundary correspondence is realized with respect to them. At the same time, if the class of mappings is selected, for example, by differential requirements, then these requirements must be sufficient for this correspondence to be provable.

This scheme of investigation in the case of plane mappings has been fully realized for a broad class of mappings and domains (works of M. A. Lavrent'ev, J. Lelong-Ferrand, and G. D. Suvorov).

We now present results that have been obtained for spatial mappings.

2°. Let the vector-function $y = T(x) \equiv [y_1(x), y_2(x), y_3(x)]$, where $x = (x_1, x_2, x_3)$, be continuously differentiable in a domain D of Euclidean space E^3 and realize a topological mapping of the domain D onto a domain $\Delta \in E^3$, with

$$I(T, D) = \iiint_D \left\{ \sum_{i,j=1}^{3} \left[\frac{\partial y_i(x)}{\partial x_j} \right]^2 \right\}^{3/2} dx_1 dx_2 dx_3 \leq k < \infty \tag{2}$$

and the integration is understood in the sense of Lebesgue.

Let ρ be the ordinary Euclidean distance. Take a point $x^0 \in D$ and introduce spherical coordinates (r, φ, ψ) with origin at x^0, $0 < r \leq R \leq \rho(x^0, \overline{D} \setminus D)$, $0 \leq \varphi \leq \pi$, $0 \leq \psi < 2\pi$. Denote by Q_r the ball $\rho(x, x^0) < r$, by S_r the sphere $\rho(x, x^0) = r$, and let $C_{r\varphi}$ be the circle of the great circle $\varphi = \text{const}$ on S_r. Then

$$\int_0^R dr \int_0^\pi L^3[T(C_{r\varphi})] \frac{d\varphi}{r} \leq c_1 I(T, Q_R), \tag{3}$$

where $L(l)$ is the length of the curve l in the Euclidean metric.

Further, if D is the open ball $\rho(x, 0) < 1$ and $x^0 \in \overline{D}$, then there is a number $\bar{r}$, $0 < r_1 \leq \bar{r} \leq r_2 < 2$, such that

$$\omega_{\bar{r}}^3(T) \ln \frac{r_2}{r_1} \leq c_2 I(T, Q_{r_2}). \tag{4}$$

Here $\omega_{\bar{r}}$ is the oscillation of the function T on the set $S_{\bar{r}} \cap \overline{D}$, and c_1 and c_2 are absolute constants.

3°. Inequalities (3) and (4) (one can also write down others of the same kind) contain information sufficient for establishing the following results (in the sense of realizing the scheme of item 1°):

Theorem 1. *If $\{T\}$ is a family of topological mappings of the unit ball D onto the unit ball Δ (under the remaining conditions of item 2°), with $T(0) = 0$ and $I(T, D) \leq k$, $I(T^{-1}, \Delta) \leq k$ for all $T \in \{T\}$, then there exist numbers $M(k)$ and $N_1(k)$ such that for $x', x'' \in D$, $\rho(x', x'') \leq M(k)$, one has:*

$$\frac{1}{2} \exp\left[-\frac{N_1(k)}{\rho^3(x', x'')}\right] \leq \rho[T(x'), T(x'')] \leq N_1^{1/3}(k) \ln^{-1/3} \frac{1}{2\rho(x', x'')} \tag{5}$$

simultaneously for all $T \in \{T\}$.

It follows from (5) that the same estimates are valid also in $\overline{D}$, and that under the mapping $T \in \{T\}$ a pointwise correspondence of boundaries is realized.

Theorem 2. *If D is the ball $\rho(x, 0) < 1$ and $\{T\}$ is a family of mappings $\Delta_T \equiv T(D)$, $T(0) = 0$, of the ball into E^3 (all the other conditions of item 2° are fulfilled, and k does not depend on $T \in \{T\}$), then for any points $x', x'' \in D$, $\rho(x', x'') < 1/2$, the estimate*

$$r_{\Delta_T}[T(x'), T(x'')] \leq N_1^{1/3}(k) \ln^{-1/3} \frac{1}{2\rho(x', x'')} \tag{6}$$

is valid for all $T \in \{T\}$. Here r_{Δ_T} is the relative distance in Δ_T, introduced by C. Mazurkiewicz (1). *

* With inessential deviations from the definition in (1), this distance can be introduced as follows: let $0 \in \Delta$ and $y', y'' \in \Delta$, $y', y'' \neq 0$, be arbitrary points; let $\mathfrak{H}(y', y'')$ be the class of point sets H with the properties: 1) $H \subset \Delta$, $0 \in H$; 2) H is connected and closed relative to Δ; 3) y' and y'' belong to the same component of connectedness of the set $\Delta \setminus H$; 4) 0 and y' belong to different components of $\Delta \setminus H$. Denote by $d(H)$ the Euclidean diameter of H. Then we put:

$$r_\Delta(y', y'') = \inf d(H),$$

where the infimum is taken over all $H \in \mathfrak{H}(y', y'')$. Just as in (1), it is verified that r_Δ satisfies the axioms of a metric space (in Δ, r_Δ is equivalent to ρ). The completion of Δ with respect to r_Δ adjoins boundary elements and the point 0.

It follows from (6) that a fundamental (in the Euclidean metric ρ) sequence of points of D passes into a fundamental (in the metric r_{Δ_T}) sequence of points of Δ_T, i.e., to a boundary point of D in Δ_T there corresponds one boundary element, and (6) is also valid in $\overline{D}^*$.

Remark 1. Under the conditions of Theorem 2, the volume of the class of admissible domains $\{\Delta\}$ remains unclear. From (6) it follows only that if $\widetilde{\Delta}$–the completion of Δ with respect to r_Δ–is not compact, then the mapping of the ball onto Δ within the selected class of mappings is impossible. But $\widetilde{\Delta}$ will fail to be compact in many cases. Let us give an example.

Let Δ' be the interior of the unit cube constructed over the square with vertices at the points $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, $(1,0,0)$, from which all points (y_1, y_2, y_3) have additionally been removed whose coordinates satisfy the conditions: $0 < y_1 < 1$, $y_2 = \dfrac{1}{2^n}$ $(n = 1, 2, ...)$, $0 < y_3 \leq \dfrac{1}{2}$. Consider the domain Δ obtained from Δ' by an arbitrary translation such that the point $(0,0,0)$ lies inside Δ. Then any sequence of points of Δ converging to a boundary point of Δ which in Δ' corresponds to a boundary point of the form $(y_1, 0, y_3)$, $0 < y_1 < 1$, $0 < y_3 < \dfrac{1}{2}$, obviously contains no subsequence converging (with respect to r_Δ) in $\widetilde{\Delta}$.

4°. Let us give two more theorems obtained by the methods of § 2°.

Theorem 3. Let $\{T\}$ be a family of topological mappings of domains $\{D_T\}$ in E^3 onto domains $\{\Delta_T\}$ of the same space: $T(D_T) = \Delta_T$. Suppose that the other conditions of § 2° are fulfilled and that all domains D_T contain inside them a certain fixed ball with center at the origin. Choose arbitrarily continua $\overline{D}'_T \subset D_T$ (T ranges over the whole family $\{T\}$) such that $\rho(\overline{D}'_T, \overline{D}_T \setminus D_T) \geq d > 0$ (d does not depend on the choice of $T \in \{T\}$).

Then for any points $x', x'' \in \overline{D}'_T$, $\rho(x', x'') < \min(3/2\, d^2, 2)$, we have

$$\rho\,[T(x'), T(x'')] \leq N_2(k)\,|\ln^{-1/3} \frac{2}{\rho(x', x'')}| \tag{7}$$

for any $T \in \{T\}$.

Estimate (7) gives the order of equicontinuity of the family $\{T\}$ inside the domains of definition in the sense of Definition 1 of paper ([3]) (p. 506).

Theorem 4. Consider a family of mappings $\{T\}$ of the ball $\rho(x, 0) < 1$ in E^3 that are monotone** (not necessarily topological) and continuously differentiable. Let $I(T, D) \leq k$ for all $T \in \{T\}$. Then the following estimate of the order of growth holds:

$$\rho\,[T(x), T(0)] < N_3(k)\,[1 - \rho(x, 0)]^{-1/3}. \tag{8}$$

5°. **Remark 2.** The classes of mappings considered include, in particular, the class of topological Q-quasiconformal mappings taking domains into domains of bounded volume, which follows from the known relation

$$\left[\sum_{i,j}^{3}\left(\frac{\partial y_i}{\partial x_j}\right)^2\right]^{3/2} \leq 3^{3/2} Q^3 \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}.$$

* The absence of a lower estimate for r_{Δ_T} does not allow one to draw a conclusion about the mutual uniqueness of the correspondence: point—boundary element. It is interesting to compare our result with the result of V. A. Zorich ([2]).

** A vector function $y = T(x)$, defined in a domain $D \subset E^3$, is called monotone if for any domain D_1, $\overline{D}_1 \subset D$, there exist points $\bar{x}', \bar{x}'' \in (\overline{D}_1 \setminus D_1)$ such that

$$\rho[T(\bar{x}'), T(\bar{x}'')] = \sup_{x', x'' \in D_1} \rho[T(x'), T(x'')].$$

Remark 3. In conclusion, let us note that in the method being developed one can:

1) Broaden the classes of domains D and Δ by introducing into consideration the spherical metric in E_3, defining relative distances on the basis of this metric, and considering the spherical analogue of the Dirichlet integral.

2) Replace the requirement of continuous differentiability by the requirement that generalized first-order partial derivatives exist in the sense of S. L. Sobolev.

Let us also note that the method admits a generalization to the n-dimensional case.

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Note: Figure translations are in progress. See original paper for figures.