MATHEMATICS

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1964

SovietRxiv

Abstract

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MATHEMATICS

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SEVERAL THEOREMS ON THE COMPLETENESS OF ROOT SUBSPACES OF COMPLETELY CONTINUOUS OPERATORS

(Presented by Academician I. G. Petrovskii on 19 XI 1963)

In this note, applying the method of estimating the resolvents of linear operators set forth in ([1]), we obtain theorems on the completeness of root subspaces for certain classes of completely continuous operators. In doing so we use both the estimates given in ([1]) and other, more special ones, which are proved by the same method.

By $\mathfrak{S}_\infty$ we shall denote the class of all completely continuous operators acting in a separable Hilbert space $\mathfrak{H}$. For each such operator A one introduces the sequence $\{s_n(A)\}_1^\infty$ of its s-numbers, defined as the eigenvalues of the operator $(A^*A)^{1/2}$, numbered in decreasing order with multiplicities taken into account, and the function $\nu(t, A)$ $(t > 0)$, giving the number of numbers $s_n(A)$ greater than $1/t$. By $\mathfrak{S}_p$ $(p > 0)$ is denoted the class of operators A $(A \in \mathfrak{S}_\infty)$ for which

$$\sum_{k=1}^{\infty} s_k^p(A) < \infty,$$

and by $\mathfrak{S}_\omega$ the class of operators A $(A \in \mathfrak{S}_\infty)$ for which

$$\sum_{k=1}^{\infty} (2k-1)^{-1} s_k(A) < \infty.$$

1. With the aid of Theorem 1 from ([1]) it is easily proved that

Theorem 1. *Let the resolvent of the operator A $(A \in \mathfrak{S}_\infty)$ admit, for $|\arg\lambda - \pi| < \pi - \pi/2\rho_1$ $(\rho_1 \geq \rho > 1/2)$, the estimate*

$$\left\|(I - \lambda A)^{-1}\right\| \leq C \sec \frac{\pi - \varphi}{2 - \rho_1^{-1}}, \qquad \lambda = re^{i\varphi}, \quad 0 \leq \varphi < 2\pi,$$

and, in addition, let the condition

$$\lim_{r\to\infty} \frac{1}{r^\rho} \int_0^r \frac{\nu(t, A)}{t}\, dt = 0$$

be satisfied.

Then the system of root subspaces of the operator A is complete in $\mathfrak{H}$. In the same assumptions, the expansions in the principal vectors of the operator A are summable by Abel's method of order α ($\rho_1 > \alpha \geq \rho$) (for the definition see ([2])).

This assertion is a generalization of a theorem of V. B. Lidskii ([2]).

The following theorem, in a somewhat weaker form, was conjectured by M. G. Krein. It generalizes the theorems of V. B. Lidskii ([3,4]), M. G. Krein ([5]), and B. Ya. Levin ([6]).

Theorem 2. *Let the operator $A = G + iH$ ($G = \frac{1}{2}(A + A^*)$) be completely continuous. If $H \geq 0$, $s_n(G^-) = o(n^{-1})$, $s_n(H) = o(n^{-1/2})$ (instead of the last equality one may require $s_n(G^+) = o(n^{-1/2})$), where G^+ and G^- are defined with the aid of the decomposition $G = G^+ + G^-$ of the operator G into the difference of two nonnegative mutually orthogonal operators, then the system of root subspaces of the operator A is complete in $\mathfrak{H}$.*

We give briefly the proof of Theorem 2. Assuming the contrary, we shall find two normalized vectors f and g such that the function $\varphi(\lambda) = ((I - \lambda A)^{-1} f, g)$ is entire, but is not identically equal to a constant. Putting in Theorem 3 from ([1]) $T = G^+$, $F = -G^- + iH$, we have

$$\ln|\varphi(\lambda)| = o\left(\frac{r^2}{\sin^4 \varphi/2}\right) \quad \text{for } 0 < \varphi < 2\pi; \qquad \frac{r}{\sin^2 \varphi/2} \to \infty. \tag{1}$$

By means of the device used by us in ([7]), it is easy to verify that from (1) there follows the equality $\ln|\varphi(\lambda)| = o(r^2)$ $(r \to \infty)$.

Let us apply once more Theorem 3 from ([1]) with $T = G^+ + iH$ and $F = -G^-$ in the angle $0 < \varphi < {}^3/_2\pi$, and obtain a new estimate for $\varphi(\lambda)$:

$$\ln|\varphi(\lambda)| = o\left(\frac{r}{\sin^{1/2}{}_{2/3}\varphi}\right) \quad \text{when } \frac{r}{\sin^{1/2}{}_{2/3}\varphi} \to \infty. \tag{2}$$

Estimate (2) can be used with the help of the following, easily proved, modification of the Phragmén-Lindelöf principle:

Lemma. *Let D be a curvilinear angle described by the inequalities*

$$|\lambda| > 1, \qquad |\varphi| < \gamma + \frac{1}{r^\alpha}, \qquad 0 < \gamma < 2\pi, \qquad 0 < \alpha < 1.$$

Let a function $f(\lambda)$, holomorphic in D and continuous in $\overline{D}$, be given in D. If on the boundary of D the inequality $|f(\lambda)| \leq 1$ holds and, moreover, everywhere in D $\ln|f(\lambda)| = o(r^{\pi/2\gamma})$ as $r \to \infty$, then the inequality $|f(\lambda)| \leq 1$ is valid everywhere in D.

Applying the formulated lemma in the curvilinear angle $-r^{-1/2} - \pi/2 < \varphi < r^{-1/2}$, and then the Phragmén-Lindelöf principle in the angle $-\pi/2-\varepsilon < \varphi < \varepsilon$, we see that $\ln|\varphi(\lambda)| = o(r)$. From the nonnegativity of the imaginary component $\hat{A}$ there follows the estimate $|\varphi(\lambda)| \leq C \operatorname{cosec}\varphi$ for $\operatorname{Im}\lambda > 0$. Finally, using the Phragmén-Lindelöf principle in the half-plane $\operatorname{Im}\lambda < 1$, we find that $\varphi(\lambda)$ is identically equal to a constant, and arrive at a contradiction. The second variant of Theorem 2 is proved analogously.

As is not hard to see, by the arguments carried out here one can also prove other theorems of this kind. We note that with such a method of proving completeness the need to use theorems on the dependence of the spectra of the Volterra components of the operator ([8–11]) disappears.

2. The known theorem of M. V. Keldysh ([12]) for the case of a linear pencil of operators ($n = 1$) is equivalent to the assertion on the completeness of the root subspaces of the operator $A = H(I+S)$, where $H = H^* \in \mathfrak{S}_p$ for some p, $S \in \mathfrak{S}_\infty$, and the operator A is annihilated only at zero. The following generalizations of this assertion can also be formulated as propositions on the completeness of the eigenvectors and associated vectors of a linear operator pencil.

Theorem 3. *Let the operators H and S be completely continuous, H self-adjoint, and let the operator $A = H(I + H^\alpha S)$ be annihilated only at zero. If the condition*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n}(2j-1)^{-1}s_j(S)}{s_n^{-\alpha}(H)} = 0, \tag{3}$$

is fulfilled, then the system of root subspaces of the operator A for $\alpha > 0$ is complete in $\mathfrak{H}$. The assertion of the theorem will also be valid for $\alpha = 0$, if the denominator in (3) is replaced by $\ln\dfrac{1}{s_n(H)}$.

Theorem 3 is proved with the help of Ahlfors' theorem on distortion under a conformal mapping ([13]) and an estimate for the resolvent of the operator A in terms of the s-numbers of the operator S.

Let us note two "extreme" cases of Theorem 3. Since the numerator in (3) is always $o(\ln n)$, the assertion of the theorem will remain valid if (3) is replaced by the following condition:

$$\begin{aligned} &\lim_{n\to\infty} \frac{\ln n}{s_n^{-\alpha}(H)} < \infty \qquad \text{for } \alpha > 0, \\ &\lim_{n\to\infty} \frac{\ln n}{\ln\left(\dfrac{1}{s_n(H)}\right)} < \infty \quad \text{for } \alpha = 0. \end{aligned} \tag{4}$$

This consequence of Theorem 3 for $\alpha = 0$* is a direct generalization of a theorem of M. V. Keldysh, for the condition $H \in \mathfrak{S}_p$ for some p is equivalent to the second of conditions (4), in which lim is replaced by $\overline{\lim}$.

From Theorem 3 there also follows** Theorem 3 of ([14]), according to which condition (3) may be replaced by the condition $H \in \mathfrak{S}_\omega$. From Theorem 3, in turn, it follows that if $H \in \mathfrak{S}_\omega$, and the operator G $(G = G^*)$ has a purely discrete spectrum (this means that the only limit point of its spectrum lies at infinity), then the operator $A = G + iH$ has a complete system of root subspaces.

It is curious that the last assertion can be reversed, namely: if the operator H $(H = H^* \in \mathfrak{S}_\infty)$ does not belong to $\mathfrak{S}_\omega$, then there exists a self-adjoint operator G with purely discrete spectrum such that the operator $A = G + iH$ has empty spectrum.

For the formulation of Theorem 4 it will be convenient for us to introduce one more notation. If the operator T belongs to the class $\mathfrak{S}_\omega$, then we define for it the sequence $\{t_n(T)\}_1^\infty$ by the equality

$$t_n(T) = \sum_{k=n}^{\infty} (2k-1)^{-1} s_k(T).$$

Theorem 4. Let the operators H and S be completely continuous, let H be self-adjoint, and let the operator $A = H(I + H^\alpha S)$ $(\alpha \geq 0)$ be annihilated only at zero. Let $H^{\alpha+1}S \in \mathfrak{S}_\omega$. Finally, suppose that the condition

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} (2k-1)^{-1} s_k(H^{\alpha+1}S)}{t_n^{-\alpha}(H^{\alpha+1}S)} = 0 \tag{5}$$

is satisfied for $\alpha > 0$, and, for $\alpha = 0$, the condition obtained from (5) by replacing the denominator by

$$\ln \frac{1}{t_n(H^{\alpha+1}S)}.$$

Then the system of root subspaces of the operator A is complete in $\mathfrak{H}$.

This theorem is proved with the aid of Warschawski' s asymptotics ([15]) for a function realizing a conformal mapping of a curvilinear half-strip, and Theorem 3 of ([1]). It can also be proved with the aid of Theorem 6 of ([10]), but the first method (although more complicated) seems to us stronger.

From Theorem 4 one obtains corollaries analogous to the corollaries from Theorem 3. In particular, from it there follows a proposition, first proved by I. Ts. Gokhberg and M. G. Krein, according to which the system of root

* This consequence was obtained from Theorem 1 of ([1]) by Yu. A. Palant and the author before Theorem 3 was proved.

** It is necessary to note that in the formulation of Theorem 3 in ([14]) the author omitted the requirement that the operator A be annihilated only at zero.

subspaces of the operator $A = H(I + S)$ is complete, if $H = H^* \in \mathfrak{S}_\infty$, $S \in \mathfrak{S}_\infty$ and $HS \in \mathfrak{S}_p$ for some p, and the operator A vanishes only at zero.

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Received
16 XI 1963

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Note: Figure translations are in progress. See original paper for figures.