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Abstract

Full Text

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ON A THEOREM OF S. N. BERNSTEIN

(Presented by Academician A. N. Kolmogorov on 8 III 1964)

In 1954 S. N. Bernstein ([1]) proved the following theorem:

Let

$$P_n(\theta) = \sum_{k=0}^{n} (a_k \cos k\theta + b_k \sin k\theta)$$

be a real trigonometric polynomial of order $\leqslant n$, and let $|P_n(\theta)| \leqslant M$. Then the polynomial

$$f_\alpha(\theta) = \sum_{k=0}^{n} [\lambda_{n-k} A_k(\theta, \alpha) + \mu_{n-k} B_k(\theta, \alpha)],$$

where

$$A_k(\theta, \alpha) = a_k \cos(k\theta + \alpha) + b_k \sin(k\theta + \alpha),$$

$$B_k(\theta, \alpha) = -a_k \sin(k\theta + \alpha) + b_k \cos(k\theta + \alpha), \qquad b_0 = \mu_0 = \mu_n = 0$$

and α is a real number, cannot, for any real value of θ, exceed (but may attain) in absolute value the quantity $\lambda_0 M$, if

$$\lambda_0 + \lambda_n \cos n\theta_\nu + 2 \sum_{k=0}^{n-1} (\lambda_k \cos k\theta_\nu - \mu_k \sin k\theta_\nu) \geqslant 0$$

for

$$\theta_\nu = \frac{\alpha + \nu\pi}{n}, \qquad \nu = 0, 1, \dots, 2n-1.$$

In the proof of this theorem methods of best approximations are used.

In the proposed note, for the polynomials $P_n(\theta)$ and

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$$Q_m(\theta) = \lambda_0 + \lambda_m \cos m\theta - \mu_m \sin m\theta + 2\sum_{k=1}^{n-1}(\lambda_k \cos k\theta - \mu_k \sin k\theta)$$

one inequality is established (see (2)), from which, in particular, the theorem of S. N. Bernstein formulated above and some other results follow.

The proof of inequality (2) is based on the following two lemmas.

Lemma 1. For any real values u, α and integers k,

$$S_1 = \sum_{\nu=0}^{\frac{2}{s}N-1} \cos k(t_\nu - u) = \begin{cases} \frac{2}{s}N \cos k\frac{Nu-\alpha}{N}, & \text{if } k = \frac{2}{s}Nj, \\ 0, & \text{if } k \neq \frac{2}{s}Nj \ (j = 0, \pm 1, ...); \end{cases}$$

$$S_2 = \sum_{\nu=0}^{\frac{2}{s}N-1} \sin k(t_\nu - u) = \begin{cases} -\frac{2}{s}N \sin k\frac{Nu-\alpha}{N}, & \text{if } k = \frac{2}{s}Nj, \\ 0, & \text{if } k \neq \frac{2}{s}Nj \ (j = 0, \pm 1, ...), \end{cases}$$

where $t_\nu = (\alpha + \nu s\pi)/N$; N and $2/S$ are natural numbers.

Lemma 2. For any real value of θ the equality

$$F_\alpha(\theta) = \frac{s}{2N} \sum_{\nu=0}^{\frac{2}{s}N-1} P_n(\theta + t_\nu)\, Q_m(t_\nu) \cos \nu(sr\pi), \tag{1}$$

holds, where

$$F_\alpha(\theta) = \frac{1}{\pi} \int_0^{2\pi} P_n(\theta + u)\, Q_m(u) \cos r(Nu - \alpha) {\sum_{j=0}^{j_0}}' \cos \frac{2}{s}j(Nu - \alpha)\, du,$$

r and j_0 are nonnegative integers satisfying the inequality $n+m+rN \leq \frac{2}{s}N(1+j_0)$ (the prime denotes that the first term of the sum is equal to 1/2).

The assertion of Lemma 1 follows directly from the identity

$$S_1 + iS_2 = \begin{cases} \frac{2}{s}Ne^{-ik\frac{Nu-\alpha}{N}}, & \text{if } k = \frac{2}{s}Nj, \\ 0, & \text{if } k \neq \frac{2}{s}Nj \ (j = 0, \pm 1, ...). \end{cases}$$

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Lemma 2 follows from the integral representation

$$P_n(\theta + t)\, Q_m(t) \cos r(Nt - \alpha) =$$

$$= \frac{1}{\pi} \int_0^{2\pi} P_n(\theta+u)\, Q_m(u) \cos r(Nu-\alpha) \sum_{k=0}^{q}{}' \cos k(t-u)\, du \qquad (q \geq n+m+rN)$$

and Lemma 1.

Applying now Minkowski' s inequality to the right-hand side of equality (1), we obtain the basic inequality

$$\|F_\alpha(\theta)\|_{L_p} \leq \frac{s}{2N} \|P_n(\theta)\|_{L_p} \sum_{\nu=0}^{\frac{2}{s}N-1} |Q_m(t_\nu) \cos \nu(sr\pi)|, \qquad 1 \leq p \leq \infty. \quad (2)$$

In the case where $Q_m(t_\nu) \geq 0$, $\nu = 0, 1, \ldots, \frac{2}{s}N - 1$, and $m < \frac{2}{s}N$,

$$\sum_{\nu=0}^{\frac{2}{s}N-1} |Q_m(t_\nu) \cos \nu(sr\pi)| \leq \frac{2}{s} N \lambda_0,$$

and inequality (2) assumes the simpler form

$$\|F_\alpha(\theta)\|_{L_p} \leq \lambda_0 \|P_n(\theta)\|_{L_p}, \qquad 1 \leq p \leq \infty. \quad (3)$$

We shall show that, when rs is an integer and $rN \leq n$, equality in (3) is attained for the polynomial $P(\theta) = c \cos r(N\theta - \alpha)$, where c is an arbitrary constant.

Indeed, since

$$F_\alpha(\theta) = \frac{s}{2N} c \sum_{\nu=0}^{\frac{2}{s}N-1} Q_m(t_\nu) \cos r[N(\theta + t_\nu) - \alpha] \cos \nu(rs\pi) =$$

$$= \frac{s}{2N} c \cos r(N\theta - \alpha) \sum_{\nu=0}^{\frac{2}{s}N-1} Q_m(t_\nu) \cos^2 \nu(rs\pi) = \lambda_0 P(\theta),$$

then

$$\|F_\alpha(\theta)\|_{L_p} = \lambda_0 \|P(\theta)\|_{L_p},$$

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as was required.

Thus, one may formulate the following theorem:

If, for a given real α, a trigonometric polynomial $Q_m(\theta)$ of order $m < \frac{2}{s}N$ satisfies the condition

$$Q_m(\theta_\nu) \geqslant 0 \left(\nu = 0, 1, \dots, \frac{2}{s}N - 1\right),$$

then, for an arbitrary trigonometric polynomial $P_n(\theta)$, the inequality

$$\|F_\alpha(\theta)\|_{L_p} \leqslant \lambda_0 \|P_n(\theta)\|_{L_p}, \qquad 1 \leqslant p \leqslant \infty,$$

holds, with equality, when rs is an integer and $rN \leqslant n$, attained for the polynomial

$$P(\theta) = c \cos r(N\theta - \alpha).$$

In particular, if $r = s = 1,\ m = N = n$, then

$$F_\alpha(\theta) = \frac{1}{\pi} \int_0^{2\pi} P_n(\theta + u) Q_n(u) \cos(nu - \alpha)\, du = f_\alpha(\theta),$$

and therefore

$$\|f_\alpha(\theta)\|_{L_p} \leqslant \lambda_0 \|P_n(\theta)\|_{L_p}, \qquad 1 \leqslant p \leqslant \infty.$$

Hence, for $p = \infty$ and $\mu_n = 0$, we obtain S. N. Bernstein' s theorem, and for

$$Q_m(\theta) = \left(\frac{\sin \frac{n-l}{2}\theta}{\sin \frac{\theta}{2}} \right)^2 = n - l + 2 \sum_{k=1}^{n-l-1} (n - l - k) \cos k\theta, \qquad n > l,$$

the inequality

$$\left\| \sum_{k=l+1}^{n} (k - l) A_k(\theta, \alpha) \right\|_{L_p} \leqslant (n - l) \|P_n\|_{L_p}, \qquad 1 \leqslant p \leqslant \infty, \tag{4}$$

which generalizes the well-known result of A. Zygmund [(2)]

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$$\|P_n'\|_{L_p} \leqslant n\|P_n\|_{L_p}. \tag{5}$$

Replacing $P_n(\theta)$ in (4) by

$$\overline{P}_n(\theta) = \sum_{k=1}^{n}(a_k \sin k\theta - b_k \cos k\theta),$$

we obtain one more inequality

$$\left\|\sum_{k=l+1}^{n}(k-l)B_k(\theta,\alpha)\right\|_{L_p} = (n-l)\|\overline{P}_n\|_{L_p};$$

in particular, for $l = 0$ and $\alpha = 0$ we have

$$\|P_n'\|_{L_p} \leqslant n\|\overline{P}_n\|_{L_p}.$$

Let now

$$\Phi_n = \sum_{k=0}^{n} \rho^k A_k(\theta,\alpha)$$

be a harmonic polynomial of order n. Put

$$\tau_l(\theta) = \sum_{k=1}^{n} \frac{k!}{(k-l)!}\rho^k A_k(\theta,\alpha) = \rho^l \frac{\partial^l \Phi_n}{\partial \rho^l}.$$

According to (4), we have:

$$\|\tau_1\|_{L_p} \leqslant n\|\Phi_n\|_{L_p},$$

$$\|\tau_2\|_{L_p} \leqslant (n-1)\|\tau_1\|_{L_p},$$

$$\|\tau_l\|_{L_p} \leqslant (n-l+1)\|\tau_{l-1}\|_{L_p}.$$

It follows from this that

$$\int_0^{2\pi} \left|\rho^l \frac{\partial^l \Phi_n}{\partial \rho^l}\right|^p d\theta \leq \left[\frac{n!}{(n-l)!}\right]^p \int_0^{2\pi} |\Phi_n|^p d\theta. \tag{6}$$

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Next, since Φ_n is a trigonometric polynomial of order n in θ, in view of (5) and (6),

$$\int_0^{2\pi} \left|\rho^l \frac{\partial^{l+k}\Phi_n}{\partial\rho^l \partial\theta^k}\right|^p d\theta \le n^{kp} \left[\frac{n!}{(n-l)!}\right]^p \int_0^{2\pi} |\Phi_n|^p d\theta.$$

Hence, for any nonnegative function $\omega(\rho)$ integrable on the interval $[a, b]$, we obtain the inequality

$$\left(\int_a^b \int_0^{2\pi} \omega(\rho) \left|\rho^l \frac{\partial^{l+k}\Phi_n}{\partial\rho^l \partial\theta^k}\right|^p d\rho\, d\theta\right)^{1/p} \le n^k \frac{n!}{(n-l)!} \left(\int_a^b \int_0^{2\pi} \omega(\rho)|\Phi_n|^p \,d\rho\, d\theta\right)^{1/p}, \tag{7}$$

$$1 \le p \le \infty,$$

where equality is attained for $\Phi_n = \rho^n A_n(\theta, \alpha)$.

Inequality (7) strengthens a result of Ya. S. Bugrov ([3]) for harmonic polynomials.

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CITED LITERATURE

[1] S. N. Bernstein, *Collected Works*, 2, Publishing House of the Academy of Sciences of the USSR, 1954, pp. 173–177.
[2] A. Zygmund, Proc. London Math. Soc., 34, 392 (1932).
[3] Ya. S. Bugrov, DAN, 115, No. 4, 639 (1957).

Note: Figure translations are in progress. See original paper for figures.

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