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Abstract

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BEST EXTREMAL MAPPINGS IN SPACE

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In this note we investigate quasiconformal mappings of domains of n-dimensional space that are closest to conformal ones.

1. Let $P^* = f(P)$ be a homeomorphic mapping of a simply connected domain G onto a domain G^*. If for any family of curves $\{C\} \subset G$ and its image $\{C^*\} \subset G^*$ we have

$$\frac{1}{K} M\{C\} \leqslant M\{C^*\} \leqslant K M\{C\}, \tag{1}$$

then the mapping is called **quasiconformal** (^{1}) ($M\{C\}$ is the modulus of the family $\{C\}$). As K in (1) it is natural to take the least of the suitable constants.

Mark on the boundary of G two continua B_0 and B_1, and let B_0^* and B_1^* be their images* under the mapping under consideration. Instead of the modulus of the family of curves joining B_0 and B_1 in G, it will be more convenient for us to consider the conformal capacity $\Gamma(G)$ of the domain G, equal to it (^{2,3}). We have

$$\frac{1}{K} \leqslant \frac{\Gamma(G^*)}{\Gamma(G)} \leqslant K.$$

Following Väisälä (^{1}), one may write

$$\inf \frac{J}{\Lambda^n} \leqslant \frac{\Gamma(G^*)}{\Gamma(G)} \leqslant \sup \frac{J}{\lambda^n}, \tag{2}$$

where λ is the least, Λ the greatest stretching at the given point, and J is the Jacobian of the mapping.

Definition 1. A quasiconformal mapping is called **best extremal** if

$$K = \frac{\Gamma(G^*)}{\Gamma(G)} \quad \text{when } \Gamma(G^*) \geqslant \Gamma(G)$$

or

$$K = \frac{\Gamma(G)}{\Gamma(G^*)} \quad \text{when } \Gamma(G^*) \leqslant \Gamma(G).$$

In other words, a mapping is called best extremal if at least one of the inequalities (2) becomes an equality.

Below the properties of best extremal mappings are clarified.

2. Let u and u^* be, respectively, the potentials of the extremal metrics (^{3}) of the domains G and G^*.

Introduce the mappings:

I. $P = \tau(Q)$, which sends any point Q on the level surface $\Sigma(P)$ of the function u (so that $u(Q) = a$) to the fixed point $P \in \Sigma(P)$.

II. $P = \varphi(Q)$, which sends any point Q on the force line $L(P)$ of the vector field ∇u to the fixed point $P \in L(P)$.

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* The mapping can be extended to the closures of G and G^*.

We introduce analogous mappings $P^* = \tau^*(Q^*)$ and $P^* = \varphi^*(Q^*)$ in the domain G^*. Further, let $P^* = T(P)$ be a homeomorphic mapping of $\Sigma(P_0)$ onto $\Sigma(P_0^*)$ such that $P_0^* = T(P_0)$, and let $P^* = F(P)$ be a homeomorphic mapping of $L(P_0)$ onto $L(P_0^*)$ such that $P_0^* = F(P_0)$; P_0 and P_0^* are fixed.

Definition 2. A mapping $Q^* = U(Q)$ of the domain G onto the domain G^* having the properties: 1) if $\varphi(Q) = P_1$, $P_1 \in \Sigma(P_0)$, and $T(P_1) = P_1^*$, then $Q^* \in {\varphi^*}^{-1}(P_1^*)$; 2) if $\tau(Q) = P_2$, $P_2 \in L(P_0)$, and $F(P_2) = P_2^*$, then $Q^* \in {\tau^*}^{-1}(P_2^*)$, is called a **special mapping**.

Consequently, in the case of a special mapping, the image of any level surface $u = \text{const}$ will be some level surface $u^* = \text{const}$, and the image of any force line of the field ∇u will be some force line of the field ∇u^*.

If the domains G and G^* are simply covered by level surfaces and their orthogonal trajectories, then the special mapping is one-to-one and is completely determined by specifying the mappings T and F.

Theorem 1. *A best extremal mapping is necessarily a special mapping.*

The proof of this theorem is based on the application of the Grötzsch principles in the form in which they are given in (^{3}), and in essence does not differ from the proof of Theorem 2 from (^{4}).

3. Let us now find out in what case a special mapping will be extremal. Put $\rho = |\nabla u|$, $\rho^* = |\nabla u^*|$. Let, for definiteness, $\Gamma(G^*) \geqslant \Gamma(G)$.

Theorem 2. *In order that a special mapping be the best extremal mapping, it is necessary and sufficient that the following conditions be satisfied simultaneously:*

1) $\lambda = \lim\limits_{Q\to P} \dfrac{|U(Q)-U(P)|}{|Q-P|}$, where $Q \in \varphi^{-1}(P)$;

2) $l'(\alpha) = \text{const}$, if we put $u(Q)=\alpha,\ u^*(Q^*)=l(\alpha)$, if $Q^*=U(Q)$;

3) $\displaystyle\int J/\lambda^n = \text{const}$.

For the proof, consider the function

$$\mu(P) = \lim_{Q\to P} \frac{|U(Q)-U(P)|}{|Q-P|}, \qquad \text{where } Q \in \varphi^{-1}(P).$$

We shall show that

$$\frac{\Gamma(G^*)}{\Gamma(G)} \leqslant \sup_{P\in G} \frac{J}{\mu^n}. \tag{3}$$

Putting

$$h(\rho) = \int_\gamma \rho\, ds, \quad \text{if } \gamma = \varphi^{-1}(P),$$

we obtain

$$h\left[\left(\frac{\rho}{\mu}\right)^*\right] = \int_{\gamma^*} \left(\frac{\rho}{\mu}\right)^* ds^* = \int_\gamma \frac{\rho}{\mu}\frac{ds^*}{ds}\, ds = \int_\gamma \rho\, ds = h(\rho), \tag{4}$$

where $\left(\dfrac{\rho}{\mu}\right)^* = \dfrac{\rho}{\mu}$, if $P^* = U(P)$.

Then from the inequality

$$\Gamma(G^*) \leqslant \frac{1}{h^n\left[(\rho/\mu)^*\right]} \int_{G^*} \left[\left(\frac{\rho}{\mu}\right)^*\right]^n dV = \frac{1}{h^n(\rho)} \int_G \rho^n \frac{J}{\mu^n}\, dV \tag{5}$$

we shall have

$$\Gamma(G^*) \le \frac{1}{h^n(\rho)} \int_G \rho^n \frac{J}{\mu^n}\, dV \le \sup \frac{J}{\mu^n} \cdot \frac{1}{h^n(\rho)} \int_G \rho^n\, dV = \Gamma(G) \sup \frac{J}{\mu^n}, \tag{6}$$

which proves (3). Since $\lambda \le \mu$, for an extremal mapping

$$\frac{\Gamma(G^*)}{\Gamma(G)} = \sup \frac{J}{\mu^n} = \sup \frac{J}{\lambda^n}. \tag{7}$$

Consequently, for the best extremality it is necessary and sufficient that the equalities in (5), (6) be realized. In (5) this is possible only in the case where $\left(\frac{\rho}{\mu}\right)^*$ is an extremal metric for G^*, whence

$$\left(\frac{\rho}{\mu}\right)^* = \text{const} \cdot \rho^*,$$

so that

$$\mu^{-1} = \text{const} \cdot \frac{\rho^*(P^*)}{\rho(P)}, \tag{8}$$

from which assertion 2) follows.

In (6) the equalities are attained only if

$$\frac{J}{\mu^n} \equiv \sup \frac{J}{\mu^n} = \text{const}, \tag{9}$$

so that

$$\sup \frac{J}{\mu^n} = \frac{J}{\mu^n} \leq \frac{J}{\lambda^n} \leq \sup \frac{J}{\lambda^n} = \sup \frac{J}{\mu^n},$$

whence property 3) and the identity $\lambda \equiv \mu$, i.e., property 1), follow. The theorem is proved.

In the case $\Gamma(G^*) \leq \Gamma(G)$, the formulation of the theorem is unchanged; one need only replace λ by Λ, and the equality in (2) will be attained on the left. To verify this, it is enough to apply Theorem 2 to the best extremal mapping in the class of mappings of G^* onto G.

If in (2) the equalities are attained on both sides at once, then

$$\frac{J}{\lambda^n} = \frac{J}{\Lambda^n} = \text{const},$$

so that $\lambda = \Lambda$, and the mapping will be conformal.

Remark. If properties 1) and 2) hold in G, and property 3) holds for some level surface $u = \alpha$, then this property holds everywhere in G.

This may be verified by considering the vector tubes of the fields ∇u and ∇u^* and using the fact that the integral $\int_{d\sigma(\alpha)} |\nabla u|^{n-1}\, dS$, where $d\sigma(\alpha)$ is the section of the vector tube by a piece of the level surface $u = \alpha$, does not depend on α (^{3}).

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Note: Figure translations are in progress. See original paper for figures.

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