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1964

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Abstract

Full Text

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NONNEGATIVE SOLUTIONS OF A BOUNDARY-VALUE PROBLEM WITH OBLIQUE DERIVATIVE

(Presented by Academician A. N. Kolmogorov on 27 III 1964)

1. Let D be a plane domain bounded by a smooth closed contour C^*, and let $v(z)$ be a Hölder-continuous vector field on C. Our aim is to study all nonnegative harmonic functions h in the domain D satisfying the boundary condition

$$\frac{\partial h}{\partial v} = 0. \tag{1}$$

More precisely, the problem is posed as follows. It is assumed that the vector $v(z)$ is tangent to the contour C only at a finite number of points. If the projection of $v(z)$ onto the outward normal to C changes sign at a point α, then the point α is called exceptional. The set of all exceptional points is denoted by Γ. We shall call solutions of problem A the harmonic functions h in the domain D that satisfy condition (1) at all points of the set $C \setminus \Gamma$. (No restrictions are imposed on the behavior of h as $z \to \alpha \in \Gamma$.) We wish to describe all nonnegative solutions of problem A.

2. Let s be the canonical parameter (arc length) on the contour C, measured from the point $\alpha \in \Gamma$ in the direction of the vector $v(\alpha)$. The point of the contour C corresponding to the value of the parameter s will be denoted by $c(s)$. Let $\theta(s)$ be the angle between $v[c(s)]$ and $c'(s)$. The function $\theta(s)$ changes sign at $s = 0$. Put $\alpha \in \Gamma_+$ if the sign changes from plus to minus, and $\alpha \in \Gamma_-$ if it changes from minus to plus. The number of points in the sets Γ_+ and Γ_- will be denoted respectively by n_+ and n_-. We shall assume that the function $\theta(s)$ has, in a neighborhood of zero, a Hölder-continuous derivative $\theta'(s)$. Put $\alpha \in \Gamma_+^0$ if $\alpha \in \Gamma_+$ and $\chi = \theta'(0) = 0$.

Theorem 1. *If $n_+ = 0$, then problem A has no nonnegative solutions other than constants. If $n_+ > 0$, then every nonnegative solution h of problem A is uniquely represented in the form*

$$h = \sum_{\alpha \in \Gamma_- \cup \Gamma_+^0} a_\alpha u_\alpha + \sum_{\alpha \in \Gamma_+^1} \left(c_\alpha^+ p_\alpha^+ + c_\alpha^- p_\alpha^-\right),$$

where $a_\alpha, c_\alpha^+, c_\alpha^-$ *are nonnegative constants,* u_α $(\alpha \in \Gamma_- \cup \Gamma_+^0)$, p_α^+, p_α^- $(\alpha \in \Gamma_+^1)$ *are certain special solutions. The solution* h *is bounded if and only if* $a_\alpha = 0$ $(\alpha \subset \Gamma_- \cup \Gamma_+^0)$.

We describe the behavior of the special solutions near exceptional points. By means of a conformal mapping one can reduce the general case to the case in which the domain D is the unit disk. In this disk consider the harmonic functions

$$\varphi_\alpha(z) = \operatorname{Im} \ln(1 - z/\alpha) = \arg(1 - z/\alpha) = \operatorname{arc\,tg}(1 - x)/y,$$

$$\psi_\alpha(z) = \operatorname{Re} \ln(1 - z/\alpha) = \ln|1 - z/\alpha| = \tfrac{1}{2} \ln\big[(1 - x)^2 + y^2\big],$$

$$\omega_\alpha(z) = \operatorname{Re}(1 - z/\alpha)^{-1} = \frac{1 - x}{(1 - x)^2 + y^2}.$$

* It is sufficient to require that the periodic function $c(t)$ $(-\infty < t < +\infty)$, which determines the contour C, have a Hölder-continuous derivative $c'(t)$.

$(x + iy = z/a)$. These functions are positive in D and continuous in $D \cup C$ everywhere except the point a. We shall agree to write $f \equiv g$ if the difference $f - g$ can be represented as the sum of a harmonic function h, continuous in $D \cup C$, and some linear combination of the functions $\varphi_\alpha(z)$ $(\alpha \in \Gamma_-)$.

Theorem 2. We have

$$u_\alpha(z) \equiv a_\alpha \left[\omega_\alpha(z) - \chi \psi_\alpha(z)\right] \qquad (\alpha \in \Gamma_- \cup \Gamma_+^0),$$

$$p_\alpha^+(z) \equiv -c_\alpha \varphi_\alpha(z), \qquad p_\alpha^-(z) \equiv c_\alpha \varphi_\alpha(z) \qquad (\alpha \in \Gamma_+),$$

where a_α and c_α are certain positive constants.

3. Martin ([1]) indicated a method that makes it possible to describe all nonnegative harmonic functions in an arbitrary domain D of Euclidean space. In order to obtain the results formulated in § 2, it was necessary to extend Martin' s method to a certain class of boundary-value problems (including problem A). As in Martin' s case, a certain compact extension E of the domain D is constructed. The domain D is everywhere dense in E. The set $B = E \setminus D$ is called the Martin boundary. To each point $b \in B$ there corresponds a nonnegative solution $k_b(z)$ of problem A. If this solution cannot be represented as the sum of two linearly independent nonnegative solutions, the point b is called minimal. Every nonnegative solution can be represented as an integral over the functions $k_b(z)$ corresponding to minimal points.

Theorem 3. Suppose $n_+ > 0$. The Martin boundary B for the boundary-value problem A decomposes into connected components B_α ($\alpha \in \Gamma$). For $\alpha \in \Gamma_-$, the component B_α consists of one minimal point b_α. The corresponding nonnegative solution is proportional to u_α. For $\alpha \in \Gamma_+$, the component B_α is a segment. The endpoints b_α^+ and b_α^- of this segment are minimal points; the corresponding solutions are proportional to p_α^+ and p_α^-. For $\alpha \in \Gamma_+^0$, some interior point b_α of the segment B_α is also minimal; the solution corresponding to it is proportional to u_α.

For $\alpha \in \Gamma_+ \setminus \Gamma_+^0$, all solutions corresponding to points of the segment B_α are expressed linearly through p_α^+ and p_α^-. For $\alpha \in \Gamma_+^0$, the solutions corresponding to points of the segment $[b_\alpha^-, b_\alpha]$ are expressed linearly through p_α^- and u_α, while the solutions corresponding to points of the segment $[b_\alpha, b_\alpha^+]$ are expressed through u_α and p_α^+.

Introduce notation for the points of the segment B_α ($\alpha \in \Gamma_+$). In the case $\alpha \in \Gamma_+ \setminus \Gamma_+^0$, denote by b_α^λ the point corresponding to the solution

$$c\big[(2+\lambda+|\lambda|)p_\alpha^+ + (2+|\lambda|-\lambda)p_\alpha^-\big]$$

(c is a constant depending on λ). In the case $\alpha \in \Gamma_+^0$, denote by b_α^λ ($\lambda \geq 0$) the point of the segment $[b_\alpha, b_\alpha^+]$ corresponding to the solution $c[u_\alpha + \lambda p_\alpha^+]$, and by $b_\alpha^{-\lambda}$ the point of the segment $[b_\alpha^-, b_\alpha]$ corresponding to the solution $c[u_\alpha + \lambda p_\alpha^-]$. (In addition, we agree to regard $b_\alpha^{+\infty} = b_\alpha^+$, $b_\alpha^{-\infty} = b_\alpha^-$.)

Let $\alpha \in \Gamma_+$, and let s be the canonical parameter introduced in § 2. Let $n(s)$ be the unit vector directed along the inward normal to the contour C at the point $c(s)$, and let $w(s,t) = c(s) + tn(s)$. Restricting the values of s and t to a sufficiently small interval $(-\varepsilon, +\varepsilon)$, we obtain a local coordinate system in some neighborhood of the point α. Put $\theta(s,t) = \theta(s)$, $\xi = 2\pi s t^{-1} |\ln(s^2+t^2)|^{-1}$ for $\alpha \in \Gamma_+ \setminus \Gamma_+^0$, and $\xi = -2\pi t^{-1}\theta(s,t)$ for $\alpha \in \Gamma_+^0$.

Theorem 4. If $\alpha \in \Gamma_-$, then for w to converge to α in the Martin topology it is necessary and sufficient that $w \to \alpha$ in the ordinary topology of the plane. If $\alpha \in \Gamma_+$, then for w to converge to α in the Martin topology it is necessary and sufficient that $w \to \alpha$ and $\xi \to \lambda$.

4. The results of § 3 are obtained by computing the Green function $g(z,w)$ of problem A and studying its behavior as $w \to \alpha \in \Gamma$.

Without loss of generality, one may assume that the domain D is the unit disk. The computation of the Green function breaks down into the following stages:

1) A pair of analytic functions $S(z)$ and $T(z)$ ($z \in D$) is constructed, regular everywhere in D, except at the point 0, where they may have poles; Hölder-continuous near the boundary C of the disk D; connected by the relation $ST = 1$ and satisfying the condition: for $z \in C$, $S(z)$ differs by a positive factor from $e^{i\theta}$, where θ is the angle between $v(z)$ and the positive direction of the tangent to C at the point z.

2) For each $\alpha \in \Gamma_+$, a bounded solution $p_\alpha(z)$ of problem A is constructed, satisfying the conditions: $p_\alpha(\alpha) = 1,\ p_\alpha(\gamma) = 0$ for $\gamma \in \Gamma_+,\ \gamma \neq \alpha$.

3) The Green function $g(z, w)$ is defined by the formula

$$g(z, w) = q(z, w) - \sum_{\alpha \in \Gamma_+} q(\alpha, w) p_\alpha(z),$$

where

$$q(z, w) = \operatorname{Re} \int_0^z T(z) z^{-1} \left[\overline{S(w)} L(z, \overline{w}^{-1}) - S(w) L(z, w)\right] dz.$$

Here, if $n_+ \geq n_-$, then

$$L(z, w) = \frac{1}{2} \frac{z + w}{z - w}.$$

If $n_- > n_+$, then we set $m = n_- - n_+$, choose some subset $\tilde{\Gamma}_-$ of the set Γ_-, consisting of $2m - 1$ points, construct for each $\gamma \in \tilde{\Gamma}_-$ the function

$$P_\gamma(w) = \gamma^{m-1} w^{1-m} \prod (w - \beta)(\gamma - \beta)^{-1}$$

(the product is taken over all $\beta \in \tilde{\Gamma}_-$ distinct from γ), and define the function $L(z, w)$ by the formula

$$L(z, w) = \frac{1}{2} \frac{z + w}{z - w} - \frac{1}{2} \sum_{\gamma \in \tilde{\Gamma}_-} P_\gamma(w) \frac{z + \gamma}{z - \gamma}.$$

5. In conclusion, let us indicate a probabilistic interpretation of some of the results described above. Consider the Wiener process in the domain D with reflection in the direction $v(z)$ at a boundary point $z \in C \setminus \Gamma$. We shall assume that the process is terminated as soon as the trajectory reaches one of the points of the set Γ. It turns out that motion starting from the point z is terminated at the point $\alpha \in \Gamma_+$ with probability $p_\alpha(z) = p_\alpha^-(z) + p_\alpha^+(z)$. The probability of reaching a point $\alpha \in \Gamma_-$ is zero. The trajectory can enter $\alpha \in \Gamma_+$ only by touching the contour C either from the positive side (the probability of this is p_α^+) or from the negative side (the probability of this is p_α^-). These probabilistic conclusions were first obtained by another method in the work of M. B. Malyutov ([2]). From Theorem 4 it is not difficult to extract also more precise information concerning the order of contact of the trajectory with the contour C.

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Received
25 III 1964

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1. R. S. Martin, Trans. Am. Math. Soc., **49**, 137 (1941).

2. M. B. Malyutov, DAN, **156**, No. 6 (1964).

Note: Figure translations are in progress. See original paper for figures.