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Abstract

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A THEORETICAL MODEL OF WEATHER FORECASTING

In most works on weather forecasting, an adiabatic model of the atmosphere in the quasigeostrophic approximation is taken as the basis ([1–4]). In recent years, a quite definite direction of research has emerged in the field of forecasting the fields of meteorological elements using the complete system of equations ([5–7]). The present work is devoted to the consideration of a theoretical model of atmospheric processes that takes into account a large number of diverse physical factors having essential significance for the formation of weather; therefore it more fully reflects the main features of the evolution of the fields of meteorological elements in a three-dimensional baroclinic atmosphere.

We shall assume that the Earth is flat and is associated with the coordinate system (x, y, p, t). Extension of the results to the case of a spherical Earth is trivial. Let u, v, τ be the components of the velocity vector, H the height of isobaric surfaces, T the temperature, q_1, q_2, q_3 the specific humidities of the gaseous, liquid, and solid phases, and J_ν the radiation intensity with frequency ν. Then the system of equations of atmospheric processes takes the following form:

$$
\begin{gathered}
\frac{\partial u}{\partial t} + (\mathbf{u}, \nabla) u - lv = -\frac{\partial H}{\partial x} + \frac{\partial}{\partial p}\lambda p^2 \frac{\partial u}{\partial p} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\
\frac{\partial v}{\partial t} + (\mathbf{u}, \nabla) v + lu = -\frac{\partial H}{\partial y} + \frac{\partial}{\partial p}\lambda p^2 \frac{\partial v}{\partial p} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \\
\frac{\partial T}{\partial t} + (\mathbf{u}, \nabla) T - \frac{\gamma_a}{g} RT \frac{\tau}{p} = \frac{\varepsilon}{c_p} + \frac{\partial}{\partial p}\lambda_{Tp}^2 \frac{\partial T}{\partial p} + \mu_T\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \tau}{\partial p} = 0, \qquad T = -\frac{p}{R}\frac{\partial H}{\partial p};
\end{gathered}
\tag{1}
$$

$$\frac{\partial q_1}{\partial t} + (\mathbf{u},\nabla) q_1 - \sum_j \alpha_{j1} q_1 = \frac{\partial}{\partial p}\lambda_{qp}^2 \frac{\partial q_1}{\partial p} + \mu_q \left(\frac{\partial^2 q_1}{\partial x^2} + \frac{\partial^2 q_1}{\partial y^2}\right),$$
$$\frac{\partial q_2}{\partial t} + (\mathbf{u},\nabla) q_2 - \sum_j \alpha_{j2} q_2 = \frac{\partial}{\partial p}\lambda_{qp}^2 \frac{\partial q_2}{\partial p} + \mu_q \left(\frac{\partial^2 q_2}{\partial x^2} + \frac{\partial^2 q_2}{\partial y^2}\right) - \delta_2 q_2, \quad (2)$$
$$\frac{\partial q_3}{\partial t} + (\mathbf{u},\nabla) q_3 - \sum_j \alpha_{j3} q_3 = \frac{\partial}{\partial p}\lambda_{qp}^2 \frac{\partial q_3}{\partial p} + \mu_q \left(\frac{\partial^2 q_3}{\partial x^2} + \frac{\partial^2 q_3}{\partial y^2}\right) - \delta_3 q_3;$$

$$\mu \frac{\partial J_\nu}{\partial z} + \alpha_\nu J_\nu = \frac{1}{4\pi}\int_0^{2\pi} d\psi' \int_{-1}^{1} d\mu' \, J_\nu g_\nu(\mu_0) + \alpha_c \eta_\nu(T); \quad (3)$$

$$\begin{gathered} \varepsilon = \varepsilon_w + \varepsilon_J, \\ \varepsilon_w = (L_{12}\alpha_{12} + L_{13}\alpha_{13}) q_1 - (L_{21}\alpha_{21} q_2 + L_{31}\alpha_{31} q_3), \\ \varepsilon_J = \int_0^\infty d\nu \, \alpha_{c\nu} \left(\int_0^{2\pi} d\psi \int_{-1}^{1} d\mu \, J_\nu - \eta_\nu\right), \end{gathered} \quad (4)$$

where $\eta_\nu(T)$ is determined by Planck' s formula, and L_{ij} is the latent heat of transitions of moisture from some forms into others. In formulas (1)–(3), in addition, the following notation has been used: α_{ij} are coefficients taking into account possible transitions of moisture from one state into another, and they are related

by the relations $\sum_j \alpha_{ij} = 0$ $(i = 1, 2, 3)$; δ_1, δ_2 are coefficients characterizing the precipitation of moisture in the form of rain and snow. It should be noted that some of the indicated coefficients are determined quite unambiguously from the laws of thermodynamics of moist-adiabatic processes, while others are determined from the processing of the results of statistical observations. The coefficients of the radiation-transfer equations $\alpha = \alpha_s + \alpha_c$ are determined through the specific humidities q_1, q_2, q_3 and the averaged characteristics of the absorbing and scattering medium. The notation for the other physical quantities in the system of equations of atmospheric processes is generally known.

In specifying the boundary conditions at the Earth' s surface $p = p_0$, we shall proceed from the following physical assumptions. Namely, at the Earth's surface the "no-slip condition" holds for the components u, v of the velocity vector. With respect to the quantity τ, we shall assume that it is given by the condition of adiabatic expansion of air, taking into account the flow around orographic obstacles and inhomogeneities. We shall take the heat flux from the soil to be zero. This condition corresponds to the natural assumption of rapid adaptation of the soil temperature to the temperature of the near-surface layer of air.

The paper also makes the assumption that the relative humidity of the air at the Earth' s surface is considered known for the entire forecast time. Such a condition replaces the relation for the water balance of the Earth–atmosphere system. Further, we shall assume that short-wave and long-wave radiation with a prescribed albedo a_ν are diffusely reflected from the Earth' s surface, taking into account the transformation of radiation and the intrinsic emission of the underlying surface, with conservation of the total flux of radiation incident on the Earth' s surface and leaving it. As for seas and oceans, at their surface the intrinsic emission will already be determined by the temperature of the surface layer of water, which is assumed to be a known function of the coordinates.

Thus, the conditions at the level of the Earth' s underlying surface are written in the form

$$u = 0, \qquad v = 0, \qquad \tau = \tau_0, \qquad q_1 = \bar{f}\bar{q}_1(T), \qquad \lambda_T p^2 \frac{\partial T}{\partial p} = 0,$$

$$J_\nu = \frac{a_\nu}{\pi} \int_0^{2\pi} d\psi \int_{-1}^{0} J_\nu \mu \, d\mu + B\eta_\nu(T) \quad (\mu > 0), \tag{5}$$

$$B = \left(\int_0^\infty \eta_\nu d\nu \right)^{-1} \int_0^\infty d\nu \, (1 - a_\nu) \int_0^{2\pi} d\psi \int_{-1}^{0} J_\nu \mu \, d\mu,$$

where $\bar{q}_1(T)$ is the maximum specific humidity. It should, however, be noted that over the sea $B = 1,\ T = T_0$, where T_0 is the prescribed temperature of the surface layer of water.

We now turn to the boundary conditions at the upper boundary of the atmosphere, at $p = 0$. We shall assume that the turbulent fluxes of momentum, heat, and water vapor at the upper boundary of the atmosphere are zero. At the same time, the specific humidities for the liquid and solid phases of moisture are assumed to be zero, and also that the only source of radiation is the Sun, emitting short-wave radiation, while the flux of long-wave radiation entering the atmosphere is zero. As a result, we arrive at the following conditions:

$$\lambda p^2 \frac{\partial u}{\partial p} = 0, \qquad \lambda p^2 \frac{\partial v}{\partial p} = 0, \qquad \lambda_T p^2 \frac{\partial T}{\partial p} = 0,$$

$$\tau = 0, \qquad \lambda_q p^2 \frac{\partial q_1}{\partial p} = 0, \qquad q_2 = q_3 = 0, \tag{6}$$

$$J_\nu = \frac{S_\nu}{2\pi} \delta(\mu - \mu_\odot) \quad (\mu < 0),$$

where S_ν is the spectrum of solar radiation, and $\mu_\odot$ is the altitude of the Sun.

As initial data we take the functions H, u, v, q_i $(i = 1, 2, 3)$. Thus the problem has been posed completely.

We now turn to the numerical solution of the problem of forecasting the fields of meteorological elements. We shall solve the problem by means of a specially defined method of splitting multidimensional operators into a sequence of one-dimensional ones ([8–15]). In the present work a further development of the ideas of the splitting method as applied to the problem of weather forecasting will be given.

Let us introduce the time interval $\Delta t = t_{j+1} - t_j$, and approximate the system of equations of the dynamics of atmospheric processes by a system of one-dimensional difference equations. To this end, we divide the full interval Δt into four equal intervals, on each of which we define the corresponding one-dimensional equations.

Thus, on the first interval we shall have

$$\frac{u^{j+1/4}-u^j}{\Delta t}+u^j\frac{\partial u^{j+1/4}}{\partial x}=\mu\frac{\partial^2 u^{j+1/4}}{\partial x^2},\qquad \frac{v^{j+1/4}-v^j}{\Delta t}+u^j\frac{\partial v^{j+1/4}}{\partial x}=\mu\frac{\partial^2 v^{j+1/4}}{\partial x^2},$$

$$\frac{T^{j+1/4}-T^j}{\Delta t}+u^j\frac{\partial T^{j+1/4}}{\partial x}=\mu_T\frac{\partial^2 T^{j+1/4}}{\partial x^2}; \tag{7}$$

on the second interval:

$$\frac{u^{j+2/4}-u^{j+1/4}}{\Delta t}+v^j\frac{\partial u^{j+2/4}}{\partial y}=\mu\frac{\partial^2 u^{j+2/4}}{\partial y^2},\qquad \frac{v^{j+2/4}-v^{j+1/4}}{\Delta t}+v^j\frac{\partial v^{j+2/4}}{\partial y}=\mu\frac{\partial^2 v^{j+2/4}}{\partial y^2}, \tag{8}$$

$$\frac{T^{j+2/4}-T^{j+1/4}}{\Delta t}+v^j\frac{\partial T^{j+2/4}}{\partial y}=\mu_T\frac{\partial^2 T^{j+2/4}}{\partial y^2};$$

on the third interval:

$$\frac{u^{j+3/4}-u^{j+2/4}}{\Delta t}+\tau^j\frac{\partial u^{j+3/4}}{\partial p}=\frac{\partial}{\partial p}\lambda p^2\frac{\partial u^{j+3/4}}{\partial p},$$

$$\frac{v^{j+3/4}-v^{j+2/4}}{\Delta t}+\tau^j\frac{\partial v^{j+3/4}}{\partial p}=\frac{\partial}{\partial p}\lambda p^2\frac{\partial v^{j+3/4}}{\partial p}, \tag{9}$$

$$\frac{T^{j+3/4}-T^{j+2/4}}{\Delta t}=\frac{\partial}{\partial p}\lambda_T p^2\frac{\partial T^{j+3/4}}{\partial p};$$

and, finally, on the last interval:

$$\frac{u^{j+1}-u^{j+3/4}}{\Delta t}-lv^{j+1}=-\frac{\partial H^{j+1}}{\partial x},\qquad \frac{v^{j+1}-v^{j+3/4}}{\Delta t}+lu^{j+1}=-\frac{\partial H^{j+1}}{\partial y},$$

$$\frac{T^{j+1}-T^{j+3/4}}{\Delta t}-\frac{\gamma_a-\gamma}{g}RT\frac{\tau^{j+1}}{p}=\frac{\varepsilon^j}{c_p},\qquad \frac{\partial u^{j+1}}{\partial x}+\frac{\partial v^{j+1}}{\partial y}+\frac{\partial \tau^{j+1}}{\partial p}=0,\quad (10)$$

$$T^{j+1}=-\frac{p}{R}\frac{\partial H^{j+1}}{\partial p}.$$

The appropriate boundary conditions must be adjoined to the system of equations (7)–(10).

The systems of equations (7)–(9) make it possible successively to obtain solutions satisfying the boundary conditions. However, obtaining a solution of the system of equations (10) presents a particular difficulty, since here the unknown function is H^{j+1}. By successive elimination of the functions u^{j+1}, v^{j+1}, T^{j+1}, and τ^{j+1}, one can arrive at an equation for the function H^{j+1}:

$$\frac{\partial}{\partial p}\frac{p^2}{m^2}\frac{\partial H^{j+1}}{\partial p}+\frac{\alpha^2}{1+\alpha^2}\left(\frac{\partial^2 H^{j+1}}{\partial x^2}+\frac{\partial^2 H^{j+1}}{\partial y^2}+\alpha_x\frac{\partial H^{j+1}}{\partial y}-\alpha_y\frac{\partial H^{j+1}}{\partial x}\right)=-f^{j+1},\quad (11)$$

where

$$f^{j+1}=\frac{\partial}{\partial p}\frac{pR}{m^2}\left(T^{j+3/4}+\alpha\frac{\varepsilon^j}{c_p l}\right)-\frac{\alpha^2}{1+\alpha^2}\left[\left(u^{j+3/4}+\alpha v^{j+3/4}\right)_x+\left(v^{j+3/4}-\alpha u^{j+3/4}\right)_y\right],\quad (12)$$

$$m^2=\frac{\gamma_a-\gamma}{l^2 g}R^2T.\quad (13)$$

Analysis of equation (11) shows that the right-hand side is a known function of the coordinates, which is determined by solving the systems of equations (7)–(9).

We now proceed to the solution of the equations for moisture transport. Analogously to the preceding case, this system can be represented in the form

$$\frac{q_i^{j+1/4}-q_i^j}{\Delta t}+u^j\frac{\partial q_i^{j+1/4}}{\partial x}=\mu_q\frac{\partial^2 q_i^{j+1/4}}{\partial x^2},\qquad \frac{q_i^{j+2/4}-q_i^{j+1/4}}{\Delta t}+v^j\frac{\partial q_i^{j+2/4}}{\partial y}=\mu_q\frac{\partial^2 q_i^{j+2/4}}{\partial y^2},$$

$$\frac{q_i^{j+3/4}-q_i^{j+2/4}}{\Delta t}+\tau^j\frac{\partial q_i^{j+3/4}}{\partial p}=\frac{\partial}{\partial p}\lambda_q p^2\frac{\partial q_i^{j+3/4}}{\partial p},\qquad \frac{q_i^{j+1}-q_i^{j+3/4}}{\Delta t}-\sum_k \alpha_{ki}q_k^{j+1}=-\delta_i q_i^{j+1}. \tag{14}$$

Here $i=1,2,3$, respectively.

Let us note that the fourth equation of system (14) is a system of three linear algebraic equations.

The systems of one-dimensional equations are solved by finite-difference methods. For this purpose, all derivatives are replaced by the corresponding central differences. As a result we arrive at three-point difference schemes, which are efficiently solved by the factorization method ([16,17]).

Equation (11) can be solved by the relaxation method in combination with the splitting method.

It can be proved that the constructed system of difference equations approximates the original differential equations with accuracy up to quantities of first order of smallness in $\alpha = l\Delta t$, and that it is stable in the mesh sense. The radiation-transfer equations are solved in the P_2-approximation by the method of spherical harmonics ([16]).

Thus, the problem of forecasting the fields of meteorological elements is decomposed into a sequence of elementary algorithms effectively realizable on computing machines.

In conclusion, let us note that if horizontal turbulent exchange is absent, then the formulated equations are still solved by the factorization method. In this case, for a correct formulation of the conditions on the boundary of the region (x, y), one should prescribe the values of the functions at points where the velocity vector is directed into the domain in which the solution is sought, as well as a condition that is a consequence of the two-point analogue of the corresponding difference equation of the system for points at which the velocity vector is directed out of the domain in which the solution is sought. This second relation will be the natural boundary condition that closes the algorithm of the corresponding problem. It should be noted, however, that application of the factorization method to the solution of the equations in the absence of turbulent exchange requires satisfaction of the Courant condition. For weather-forecasting problems, as is known, this condition is not a significant limitation.

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Note: Figure translations are in progress. See original paper for figures.