Corresponding Member of the USSR Academy of Sciences V. V. SOBOLEV

1964

SovietRxiv

Abstract

Full Text

PHYSICS

Corresponding Member of the USSR Academy of Sciences V. V. SOBOLEV

DIFFUSION OF RADIATION IN A PLANE LAYER OF LARGE OPTICAL THICKNESS

In the author' s notes ([1,2]), problems on the diffusion of radiation in a semi-infinite medium and in a plane layer of finite optical thickness τ_0 were considered. We shall now assume that the optical thickness of the layer is very large ($\tau_0 \gg 1$). In this case asymptotic formulas will be found for the principal quantities characterizing the radiation field in the layer (the more accurate, the larger τ_0 is).

We shall assume that in an elementary volume of the layer isotropic scattering of radiation occurs, and that the ratio of the scattering coefficient to the sum of the coefficients of scattering and true absorption is equal to λ. As is known, the problem of radiation diffusion in a layer reduces to the following integral equation for the function $B(\tau, \tau_0)$:

$$B(\tau, \tau_0) = \frac{\lambda}{2} \int_0^{\tau_0} \operatorname{Ei} |\tau - \tau'| \, B(\tau', \tau_0) \, d\tau' + g(\tau), \tag{1}$$

where the function $g(\tau)$ characterizes the location and strength of the radiation sources.

Earlier ([2]) it was shown that the resolvent of this equation $\Gamma(\tau, \tau', \tau_0)$ is expressed in terms of the function $\Phi(\tau, \tau_0) = \Gamma(0, \tau, \tau_0)$. Therefore, in order to obtain the solution of equation (1) for any function $g(\tau)$, it is sufficient to know the function $\Phi(\tau, \tau_0)$, which is determined by the equation

$$\Phi(\tau, \tau_0) = \frac{\lambda}{2} \int_0^{\tau_0} \operatorname{Ei} |\tau - \tau'| \, \Phi(\tau', \tau_0) \, d\tau' + \frac{\lambda}{2} \operatorname{Ei} \tau. \tag{2}$$

For small values of τ_0 the function $\Phi(\tau, \tau_0)$ was tabulated ([3]). We shall now obtain asymptotic formulas for the function $\Phi(\tau, \tau_0)$ for $\tau_0 \gg 1$. In doing so we shall assume that the function $\Phi(\tau)$, which represents the function $\Phi(\tau, \tau_0)$ for $\tau_0 = \infty$, is known. The equations and formulas determining the function $\Phi(\tau)$ were given earlier ([1,4]).

Let $p(\tau, \eta, \tau_0)\, d\omega$ be the probability that a quantum absorbed at optical depth τ will emerge from the layer after diffusion in it at an angle $\arccos \eta$ to the inward normal within the solid angle $d\omega$. We denote the analogous probability for a semi-infinite medium by $p(\tau,\eta)\,d\omega$. To relate the quantities $p(\tau,\eta,\tau_0)$ and $p(\tau,\eta)$ to each other, let us mentally cut the semi-infinite medium at optical depth τ_0. It is easy to obtain that

$$p(\tau,\eta) = p(\tau,\eta,\tau_0) + 2\pi \int_{\tau_0}^{\infty} p(\tau',\eta)\,d\tau' \int_0^1 p(\tau_0-\tau,\xi,\tau_0) e^{-(\tau'-\tau_0)/\xi} \frac{d\xi}{\xi}. \quad (3)$$

As is known ([5]), for $\tau \gg 1$ we have

$$p(\tau,\eta) = \frac{A}{\pi} \frac{\varphi(\eta)\eta}{1-k\eta} e^{-k\tau}, \quad (4)$$

where k is determined from the equation $\dfrac{\lambda}{2k} \ln \dfrac{1+k}{1-k} = 1$, and

$$2A \int_0^1 \frac{\varphi(\eta)\eta}{(1-k\eta)^2}\,d\eta = 1. \quad (5)$$

Here $\varphi(\eta)$ is the function introduced into the theory by V. A. Ambartsumian [(6)] and studied in detail by Chandrasekhar [(7)].

Substitution of (4) into (3) gives

$$p(\tau,\eta,\tau_0) = p(\tau,\eta) - 2\pi p(\tau_0,\eta) \int_0^1 p(\tau_0-\tau,\xi,\tau_0) \frac{d\xi}{1+k\xi}. \quad (6)$$

Introducing the notation

$$D(\tau,\tau_0) = 2\pi \int_0^1 \frac{p(\tau,\xi,\tau_0)\,d\xi}{1+k\xi}, \qquad D(\tau) = 2\pi \int_0^1 \frac{p(\tau,\xi)\,d\xi}{1+k\xi}, \quad (7)$$

from (6) we obtain

$$D(\tau,\tau_0) = \frac{D(\tau) - D(\tau_0)D(\tau_0-\tau)}{1-D^2(\tau_0)}, \quad (8)$$

where, on the basis of (7) and (4),

$$D(\tau_0) = 2Ae^{-k\tau_0} \int_0^1 \frac{\varphi(\xi)\xi}{1-k^2\xi^2}\,d\xi. \quad (9)$$

Substituting (8) into (6) and taking into account the relations

$$\Phi(\tau,\tau_0) = 2\pi \int_0^1 p(\tau,\eta,\tau_0)\frac{d\eta}{\eta}, \qquad \Phi(\tau) = 2\pi \int_0^1 p(\tau,\eta)\frac{d\eta}{\eta}, \tag{10}$$

we have

$$\Phi(\tau,\tau_0) = \Phi(\tau) - \Phi(\tau_0)\frac{D(\tau_0-\tau) - D(\tau_0)D(\tau)}{1-D^2(\tau_0)}, \tag{11}$$

where

$$\Phi(\tau_0) = \frac{4A}{\lambda}e^{-k\tau_0}. \tag{12}$$

To determine the function $D(\tau)$, we shall use the equation

$$\frac{\partial p(\tau,\eta)}{\partial \tau} = -\frac{1}{\eta}p(\tau,\eta) + p(0,\eta)\Phi(\tau), \tag{13}$$

where $p(0,\eta) = \dfrac{\lambda}{4\pi}\varphi(\eta)$ (see (5)). From (13), (10), and (7) we find

$$D(\tau) = \left(1 - \frac{\lambda}{2}\int_0^1 \frac{\varphi(\eta)\,d\eta}{1+k\eta}\right)\int_\tau^\infty \Phi(\tau')e^{k(\tau-\tau')}\,d\tau'. \tag{14}$$

Thus, for $\tau_0 \gg 1$, the desired function $\Phi(\tau,\tau_0)$ is expressed in terms of the function $\Phi(\tau)$ by formula (11), in which the quantity $D(\tau)$ is given by formula (14).

The following two special cases are of interest.

1) The role of true absorption in the layer is large ($k\tau_0 \gg 1$). In this case, instead of (11), we have

$$\Phi(\tau,\tau_0) = \Phi(\tau) - \Phi(\tau_0)\,[D(\tau_0-\tau) - D(\tau_0)D(\tau)]. \tag{15}$$

Near the upper boundary of the layer (i.e., for small τ), using formula (14), we obtain

$$\Phi(\tau,\tau_0) = \Phi(\tau) - \Phi(\tau_0)D(\tau_0)\left[e^{k\tau} - D(\tau)\right], \tag{16}$$

and near the lower boundary of the layer (i.e., for τ close to τ_0):

$$\Phi(\tau,\tau_0) = \Phi(\tau) - \Phi(\tau_0)D(\tau_0-\tau). \tag{17}$$

2) The role of true absorption in the layer is small ($k\tau_0 \ll 1$). In particular, this case occurs when $\lambda = 1$, and $k = 0$. Using the smallness of k, from (9) and (5) we find

$$D(\tau_0) = 1 - k(\tau_0 + \gamma), \tag{18}$$

where $\gamma = 2a_2/a_1$ for $\lambda = 1$, and a_n is the n-th moment of the function $\varphi(\eta)$.

To obtain an expression for $D(\tau)$ for small k, we rewrite formula (14) in the form

$$D(\tau) = D(0)e^{k\tau} - [1 - D(0)] \int_0^\tau \Phi(\tau')e^{k(\tau-\tau')}\, d\tau', \tag{19}$$

where

$$D(0) = \frac{\lambda}{2} \int_0^1 \frac{\varphi(\eta)}{1 + k\eta}\, d\eta. \tag{20}$$

For small k we have

$$D(0) = 1 - \frac{2}{\sqrt{3}}\, k. \tag{21}$$

Taking also into account that, for $k = 0$,

$$\int_0^\tau \Phi(\tau')\, d\tau' = [\tau + q(\tau)]\sqrt{3} - 1, \tag{22}$$

where $q(\tau)$ is Hopf’ s function (see [1]), from (19) we obtain

$$D(\tau) = 1 - k[\tau + 2q(\tau)]. \tag{23}$$

With the aid of (18) and (23), formula (11) is brought to the form

$$\Phi(\tau, \tau_0) = \Phi(\tau) - \frac{\tau + q(\tau) - q(\tau_0 - \tau) + \gamma/2}{\tau_0 + \gamma}\sqrt{3}, \tag{24}$$

where it has been taken into account that $\Phi(\tau_0) = \sqrt{3}$ and $q(\tau_0) = \gamma/2$. Formula (24) is the desired one for the case $k\tau_0 \ll 1$.

Near the upper boundary of the layer, from (24) we have

$$\Phi(\tau, \tau_0) = \Phi(\tau) - \frac{\tau + q(\tau)}{\tau_0 + \gamma}\sqrt{3}, \tag{25}$$

and near the lower boundary

$$\Phi(\tau, \tau_0) = \frac{\tau_0 - \tau + q(\tau_0 - \tau)}{\tau_0 + \gamma}\sqrt{3}. \tag{26}$$

With the aid of the formulas obtained, one can easily find asymptotic expressions for Ambartsumian' s functions $\varphi(\eta, \tau_0)$ and $\psi(\eta, \tau_0)$ (they are also called Chandrasekhar' s functions $X(\mu, \tau_0)$ and $Y(\mu, \tau_0)$). The author [2] gave the following formulas expressing these functions through the function $\Phi(\tau, \tau_0)$:

$$\varphi(\eta, \tau_0) = 1 + \int_0^{\tau_0} \Phi(\tau, \tau_0) e^{-\tau/\eta}\, d\tau, \tag{27}$$

$$\psi(\eta, \tau_0) = e^{-\tau_0/\eta} + \int_0^{\tau_0} \Phi(\tau_0 - \tau, \tau_0) e^{-\tau/\eta}\, d\tau. \tag{28}$$

Substituting (11) into (27) and (28), for $\tau_0 \gg 1$ we find

$$\varphi(\eta, \tau_0) = \varphi(\eta) - \frac{Ce^{-2k\tau_0}}{1 - \dfrac{C}{2k}e^{-2k\tau_0}} \frac{\eta}{1 - k\eta}\, \varphi(\eta), \tag{29}$$

$$\psi(\eta, \tau_0) = \frac{C_1 e^{-k\tau_0}}{1 - \dfrac{C}{2k}e^{-2k\tau_0}} \frac{\eta}{1 - k\eta}\, \varphi(\eta), \tag{30}$$

where

$$2kC = C_1^2, \tag{31}$$

$$C_1 \int_0^1 \frac{\varphi(\xi)\xi\, d\xi}{(1 - k\xi)^2} = 2k \int_0^1 \frac{\varphi(\xi)\xi\, d\xi}{1 - k^2\xi^2}. \tag{32}$$

Here we have used formulas (14), (12), (9), and (5).

From (29) and (30) follow the asymptotic formulas for the functions $\varphi(\eta, \tau_0)$ and $\psi(\eta, \tau_0)$, obtained earlier by another method (8). Namely, in the case $k\tau_0 \gg 1$ we have

$$\varphi(\eta, \tau_0) = \varphi(\eta) - Ce^{-2k\tau_0} \frac{\eta}{1 - k\eta} \varphi(\eta), \tag{33}$$

$$\psi(\eta, \tau_0) = C_1 e^{-k\tau_0} \frac{\eta}{1 - k\eta} \varphi(\eta), \tag{34}$$

and in the case $k\tau_0 \ll 1$

$$\varphi(\eta, \tau_0) = \varphi(\eta) - \frac{\eta\varphi(\eta)}{\tau_0 + \gamma}, \tag{35}$$

$$\psi(\eta, \tau_0) = \frac{\eta\varphi(\eta)}{\tau_0 + \gamma}. \tag{36}$$

The author expresses gratitude to V. V. Ivanov for a useful discussion of this work.

Leningrad State University
named after A. A. Zhdanov

Received
19 XI 1963

REFERENCES

[1] V. V. Sobolev, *Dokl. Akad. Nauk SSSR*, **116**, No. 1 (1957).
[2] V. V. Sobolev, *Dokl. Akad. Nauk SSSR*, **120**, No. 1 (1958).
[3] V. V. Sobolev, I. N. Minin, *Astron. Zh.*, **38**, No. 6 (1961).
[4] I. N. Minin, *Dokl. Akad. Nauk SSSR*, **120**, No. 1 (1958).
[5] V. V. Sobolev, *Transfer of Radiant Energy in the Atmospheres of Stars and Planets*, Moscow, 1956.
[6] V. A. Ambartsumian, *Astron. Zh.*, **19**, No. 5 (1942); *Nauchn. tr.*, **1**, Yerevan, 1960.
[7] S. Chandrasekhar, *Radiative Transfer*, IL, 1953.
[8] V. V. Sobolev, *Astron. Zh.*, **34**, No. 3 (1957).

Note: Figure translations are in progress. See original paper for figures.