THE SPACE OF RELATIVE TOPOLOGY*

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Abstract

Full Text

Mathematics

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THE SPACE OF RELATIVE TOPOLOGY*

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In (4) and independently, but somewhat later, in (5)**, a topologically invariant concept was defined that generalizes the concept of a space of relative metric of Mazurkiewicz-Urysohn (see (1), p. 559, and (2), p. 171). Below we give the definition of this concept and state a number of assertions established in studying it.

1. Let X_ρ be a bounded metric space***. C. Mazurkiewicz calls the *relative distance* between two of its points the lower bound of the diameters of connected sets containing these points, if such sets exist, and the diameter of X_ρ itself otherwise. The relative distance turns the set X into a new metric space $X_{\tilde{\rho}}$—the space of the relative metric X_ρ.

It is easy to give the concept of a space of relative metric a topologically invariant character. Namely, let X_τ be a topological space, $g = \{G_\alpha\}$ its base. It turns out that the system $\tilde{g}$, consisting of the components of the elements of this base, generates on the set X some (generally speaking, new) topology and that this topology does not depend on the choice of the base g of the original space X_τ. The topological space thus obtained will henceforth be denoted by $X_{\tilde{\tau}}$ and will be called the **space of relative topology** of the space X_τ. Its connection with the space of relative metric is described by

Theorem 1. *If X_ρ is a bounded metric space, then the identity mapping*

$$\xi : X_{\tilde{\tau}(\rho)} \to X_{\tau(\tilde{\rho})}$$

is a homeomorphism.

2. Let us turn to the study of the space $X_{\tilde{\tau}}$. It is easily verified that the identity mapping $\xi : X_{\tilde{\tau}} \to X_\tau$ is a homeomorphism if and only if X_τ is locally connected. (This mapping is, obviously, always continuous.)

Theorem 2. *If X_τ is Hausdorff (regular, completely regular), then $X_{\tilde{\tau}}$ is also Hausdorff (regular, completely regular)****.*

Theorem 3***.** *If X_τ is Hausdorff, then $X_{\tilde{\tau}}$ is bicompact if and only if X_τ is a locally connected bicompactum.*

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* Reported (except for item 3) at the All-Union Topological Conference on 30 IX 1963.

** I learned about paper (4) from Z. Frolík after my report at the topological conference (and, consequently, already after the publication of (5)). A substantial part of the overlap of results is contained in Theorems 1 and 4.

*** The letters X and Y denote sets. The assignment on any one of them of a metric (or topological) structure is indicated by the lower index ρ (or τ). Different metric (topological) structures of one and the same set are distinguished by means of symbols attached to these indices. (For example: X_ρ, $X_{\tilde{\rho}}$, or Y_{τ_1}, Y_{τ_2}, etc.). If X_ρ is a metric space, then the topological space generated by it is denoted by $X_{\tau(\rho)}$.

**** It is unknown to me whether normality of X_τ implies normality of $X_{\tilde{\tau}}$. Metrizability of $X_{\tilde{\tau}}$ follows from metrizability of X_τ by virtue of Theorem 1.

***** For the space of relative metric of the continuum an analogous assertion was proved by P. S. Urysohn (see (1), p. 560).

Very useful is

Theorem 4. If the mapping $f : X_{\tau_1} \to Y_{\tau_2}$ is continuous, then the mapping $f : X_{\tilde{\tau}1} \to Y_{\tilde{\tau}2}$ is also continuous.

Corollary 1. If $X_\tau \subseteq Y_\tau$, then the identity mapping $\xi : X_{\tilde{\tau}} \to Y_{\tilde{\tau}}$ is continuous.

3. Definition 1. A subspace X_τ of a space Y_τ is called **properly situated** in Y_τ if the identity mapping $\xi : X_{\tilde{\tau}} \to Y_{\tilde{\tau}}$ is a homeomorphism.

For what follows the following are convenient.

Notation. Let $x \in A \subseteq X_\tau$. By $c_x^\tau A$ (respectively $k_x^\tau A$) we denote the component (respectively, quasicomponent) of the set A containing the point x.

Theorem 5. A subspace X_τ of a space Y_τ is properly situated in Y_τ if and only if, for every point $x \in X_\tau$ and every neighborhood $U \subseteq Y_\tau$ of it, there exists a smaller neighborhood $V \subseteq Y_\tau$ of it such that

$$X_\tau \cap (c_x^\tau V) \subseteq c_x^\tau (X_\tau \cap U).$$

Corollary 2. The following subsets of a topological space are properly situated in it: a) open subsets, b) locally connected subsets, c) components of open sets.

Theorem 6. A completely regular space X_τ is properly situated in at least one of its bicompact extensions* if and only if, for every point $x \in X_\tau$ and every neighborhood U of it, there exists a smaller neighborhood V of it such that

$$k_x^\tau V \subseteq c_x^\tau U.$$

Let us note that if a completely regular space is properly situated in the smaller of two of its bicompact extensions, then it is properly situated also in the larger of them. In this connection the following is of interest.

Theorem 7. Among all those bicompact extensions of a completely regular space X_τ in which it is properly situated, there exists a minimal one if and only if X_τ is locally bicompact. In this case X_τ is properly situated in all of its bicompact extensions.

4. In the study of properties of local connectedness and local bicompactness, the following plays an essential role.

Condition (χ)**.** A space X_τ is said, by definition, to satisfy condition (χ) at a point $x \in X_\tau$ if there exists a neighborhood U of this point such that the set $c_x^\tau U$ is locally connected and locally bicompact.

Theorem 8. If a space X_τ satisfies condition (χ) at every point, then $X_{\tilde{\tau}}$ is locally connected**.

The converse assertion does not hold.

In Theorem 9 and Corollary 3 the space X_τ is assumed to be Hausdorff.

Theorem 9. The space $X_{\tilde{\tau}}$ is locally bicompact if and only if X_τ satisfies condition (χ) at every point.

Corollary 3. If $X_{\tilde{\tau}}$ is locally bicompact, then it is locally connected.

5. Let X_τ be bicompact. As was already said, the identity mapping $\xi : X_{\tilde{\tau}} \to X_\tau$ is continuous. The existence of such bicompact extensions of the space $X_{\tilde{\tau}}$ (we shall call them **branched**) to which this mapping can be continuously extended is easily proved. It is also clear that if the smaller of two bicompact extensions

* Bicompact extensions are everywhere assumed to be Hausdorff.

** In fact, this assertion follows only from the first part of condition (χ), namely from the local connectedness of U (without the assumption that this set is locally bicompact).

extensions of the space $X_{\tilde{\tau}}$ is branched, then the larger one is also branched. Therefore the following is legitimate.

Definition 2. A bicompactum X_τ is called **branching** if, among all branched bicompact extensions of the space $X_{\tilde{\tau}}$, there exists a minimal one.

Theorem 10. *If the bicompactum X_τ satisfies condition (χ) at every point, then it is branching.**

Simple examples show that condition (χ) is not necessary in order that the bicompactum X_τ be branching. To formulate a necessary and sufficient criterion we introduce the following

Condition (K). The space X_τ **satisfies**, by definition, **condition (K) at a point** $x \in X_\tau$ if for every neighborhood U of this point there exists a smaller neighborhood V of it such that the set $c_x^\tau V$ is locally connected at all points**

belonging to the intersection
$(c_x^\tau V) \cap \{c_x^\tau U \setminus \langle c_x^\tau U \rangle\}$.

It is clear that condition (K) follows from condition (χ).

Theorem 10^*. *A bicompactum X_τ is branching if and only if it satisfies condition (K) at every point of local non-connectedness.*

6. If X_τ is a completely regular space, then the study of the space $X_{\tilde{\tau}}$ can be approached from the point of view of uniform structures***.

Suppose that the set X is endowed with a uniform structure Σ**** (see ([3]), p. 562). The uniform and topological spaces generated by this structure will be denoted respectively by X_Σ and $X_{\tau(\Sigma)}$. (As is known from ([3]), $X_{\tau(\Sigma)}$ is completely regular.)

To each covering $\sigma \in \Sigma$ we assign the covering $\tilde{\sigma}$, consisting of the components***** of the elements of σ, and consider the system $\widetilde{\Sigma}$ of all such coverings, into each of which a covering of the form $\tilde{\sigma}$ (where $\sigma \in \Sigma$) can be inscribed.

Theorem 11. *The system $\widetilde{\Sigma}$ forms a separated uniform structure.*

Thus a new uniform space $X_{\widetilde{\Sigma}}$ arises. The connection of the completely regular topological space $X_{\tau(\widetilde{\Sigma})}$ generated by it with the space of relative topology is established by

Theorem 12. *The identity mapping*
$\xi : X_{\tau(\widetilde{\Sigma})} \to X_{\widetilde{\tau(\Sigma)}}$ *is a homeomorphism******.*

Corollary 4. *If the mapping*
$f : X_{\tau(\Sigma_1)} \to Y_{\tau(\Sigma_2)}$ *is continuous, then the mapping*
$f : X_{\tau(\widetilde{\Sigma}1)} \to Y_{\tau(\widetilde{\Sigma}2)}$ *is also continuous.*

Corollary 5. *If the uniform structures Σ_1 and Σ_2 on the set X generate on it one and the same topology, then $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ also generate on X one and the same topology.*

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* By virtue of Theorem 9, this assertion follows from the following general fact: a completely regular space Y_τ is locally bicompact if and only if, under any continuous mapping of it into any bicompactum, there exists a minimal one among all those bicompact extensions of Y_τ to which this mapping can be continuously extended.

** A space Y_τ is called locally connected at a point $y \in Y_\tau$ if, for every neighborhood U of this point, the inclusion $y \in \langle c_y^\tau U \rangle$ holds (see ([2]), p. 171).

*** Yu. M. Smirnov drew my attention to the possibility of such an approach. I take this opportunity to note that conversations with him were very useful to me in my work.

**** Only separated uniform structures are considered.

***** In the topology of the space $X_{\tau(\Sigma)}$.

****** Since every completely regular space is generated by some uniformity (see ([3])), this also implies the part of the assertion of Theorem 2 that concerns completely regular spaces.

An analogue of Theorem 4 is

Theorem 13. *If the mapping $f : X_{\Sigma_1} \to Y_{\Sigma_2}$ is uniformly continuous, then the mapping $f : X_{\widetilde{\Sigma}1} \to Y_{\widetilde{\Sigma}2}$ is also uniformly continuous.*

7. Consider the δ-space X_δ (see (3)) and some uniform structure Σ of it. Then the δ-space $X_{\delta(\widetilde{\Sigma})}$ is uniquely determined (by analogy with $X_{\tau(\widetilde{\Sigma})}$). In particular, $X_{\delta(\widetilde{\Sigma}^0)}$ is determined, where Σ^0 is a precompact structure of the space X_δ. If X_δ has a maximal structure Σ^∞, then $X_{\delta(\widetilde{\Sigma}^\infty)}$ is also determined. In what follows we shall denote the spaces $X_{\delta(\widetilde{\Sigma}^0)}$ and $X_{\delta(\widetilde{\Sigma}^\infty)}$ respectively by X_{δ^0} and X_{δ^∞}.

A natural question arises: if Σ_1 and Σ_2 are uniform structures of one and the same δ-space X_δ, do $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ generate the same closeness on the set X? In other words, is an analogue of Corollary 5 (or of the more general Corollary 4) true for δ-spaces?

It turns out that it is not; namely: *there exists a metrizable space of countable weight X_ρ such that the structures $\widetilde{\Sigma}^0$ and $\widetilde{\Sigma}^\infty$ generate different closenesses on the set X.*

Nevertheless, from Theorem 12 and Corollary 5 it follows that

Theorem 14. *If Σ_1 and Σ_2 are uniform structures of one and the same δ-space X_δ, then $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ generate on X one and the same topology, coinciding with the topology of each of the spaces $X_{\tau(\widetilde{\Sigma}1)}$ and $X_{\tau(\widetilde{\Sigma}2)}$.*

The following assertions are also true:

Theorem 15. *If $f : X_{\delta_1} \to Y_{\delta_2}$ is a δ-mapping, and Σ_1 is an arbitrary uniform structure of X_{δ_1}, then $f : X_{\delta(\widetilde{\Sigma}1)} \to Y_{\delta_2^0}$ is also a δ-mapping.*

Corollary 6. *If $f : X_{\delta_1} \rightleftarrows Y_{\delta_2}$ is a δ-homeomorphism, then $f : X_{\delta_1^0} \rightleftarrows Y_{\delta_2^0}$ is also a δ-homeomorphism.*

For the spaces $X_{\delta_1^\infty}$ and $Y_{\delta_2^\infty}$ (if they exist) analogues of Theorem 15* and Corollary 6 hold.

Theorem 16. *If X_ρ is a bounded metrizable space, then the closeness generated on X by the metric of the space X_ρ coincides with the closeness of the space $X_{\delta(\widetilde{\Sigma}^\infty)}$, where Σ^∞ is the metrizable** structure of X_ρ.*

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CITED LITERATURE

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4. Z. Frolik, *Czechoslovak Mathematical Journal*, 11 (86), No. 3 (1961).
5. O. V. Lokutsievskii, *DAN*, 151, No. 4 (1963).

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* The δ-mapping turns out to be $f : X_{\delta_1^\infty} \to Y_{\delta(\widetilde{\Sigma}2)}$, where Σ_2 is an arbitrary uniform structure of Y_{δ_2}.

** That is, maximal.

Note: Figure translations are in progress. See original paper for figures.