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Abstract

Full Text

MATHEMATICS

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RELATION BETWEEN WEIGHTED NORMS OF AN ENTIRE FUNCTION OF FINITE DEGREE ON LINES PARALLEL TO THE REAL AXIS

(Presented by Academician I. M. Vinogradov on 19 II 1964)

Let $f(z)$ be an entire function of finite degree σ, and let $\varphi(x) \geqslant 1$ be a continuous function on the entire real axis. Along with the already considered classes $B_\sigma^{(\varphi)}$ and $W_\sigma^{(p,\varphi)}$ of entire functions of finite degree (see ([1]), pp. 37-38), we introduce into consideration also the class ${}_\sigma^{(p,\varphi)}$ of entire functions $f(z)$ of finite degree σ satisfying the conditions:

1) $|f(x+iy)| \leqslant |f(x-iy)|$;
2)

$$\|f\|_{p,\varphi} = \left(\int_{-\infty}^{\infty} \left| \frac{f(x)}{\varphi(x)} \right|^p dx \right)^{1/p} < +\infty \quad (p \geqslant 1).$$

In the case $\varphi \equiv 1$, the classes $B_\sigma^{(1)}$, ${}_\sigma^{(p,1)}$, and $W_\sigma^{(p,1)}$ are denoted respectively by B_σ, ${}_\sigma^{(p)}$, and $W_\sigma^{(p)}$.

In the present note, first, for entire functions $f(z)$ from the class $W_\sigma^{(p,\varphi)}$ an estimate is given for the norm

$$\left\| \frac{1}{\varphi(x)} \left[f(x+iy)e^{-i\omega} + f(x-iy)e^{i\omega} \right] \right\|_p = \left\| f(x+iy)e^{-i\omega} + f(x-iy)e^{i\omega} \right\|_{p,\varphi},$$

where ω is a real parameter, in terms of the norm $\|f\|_{p,\varphi}$, which is of auxiliary character. In the special case $\varphi \equiv 1$, this problem was considered by P. Boas ([2]).

Secondly, for a function $f(z) \in {}_\sigma^{(p,\varphi)}$, a relation is found between the norms $\|f(x+iy)\|_{p,\varphi}$ and $\|f(x)\|_{p,\varphi}$ in the form of an inequality. This problem was also considered in the special case, when $\varphi \equiv 1$, by P. Boas and K. Rahman ([3]).

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Further, for a function $f(z) \in {}^{(p,\varphi)}_{\sigma}$, an inequality of S. M. Nikol' skii type is established; namely, an inequality is found between the norms $\|f(x+iy)\|_{p',\varphi}$ and $\|f(x)\|_{p,\varphi}$, where $1 \leqslant p < p' \leqslant \infty$.

For the solution of the indicated problems, the following conditions are imposed on the weight function $\varphi(x)$:

$$\alpha_{\varphi}(t) = \sup_{\substack{-\infty<x<\infty \\ |y|\leqslant t}} \frac{\varphi(x+y)}{\varphi(x)} \leqslant P_m(t) = \sum_{k=0}^{m} A_k t^k, \tag{1}$$

where $A_k \geqslant 0$ $(k = 0, 1, \dots, m)$ are the coefficients of the polynomial $P_m(t)$. In addition, the notation

$$M = \sum_{k=0}^{m} A_k$$

is used.

1. Let $f(z) \in W_{\sigma}^{(p,\varphi)}$, and consider the auxiliary function*

$$g(z) = f(z)\left[\frac{\sin(z-t)}{z-t}\right]^{m}, \tag{2}$$

* In the papers ([1,5]), in the expression $g(z)$, instead of $z-t$ one took $\lambda(z-t)$, where $\lambda > 0$ is a real number. We have observed that, without diminishing the generality of the arguments carried out in solving similar problems, one may always assume $\lambda = 1$.

where $m > 0$ is an integer, and t is a real parameter. It turns out that, for the function $g(z)$, the inequalities

$$\|g\|_C = \sup_{-\infty<x<\infty} |g(x)| \leq M\varphi(t)\|f\|_{C,\varphi}, \tag{3}$$

$$\|g\|_p = \|g(x)\|_p \leq M\varphi(t)\|f\|_{p,\varphi} \tag{4}$$

hold for every fixed t, where

$$\|f\|_{C,\varphi} = \sup_{-\infty<x<\infty} \left|\frac{f(x)}{\varphi(x)}\right|.$$

It follows from this that P. Boas' interpolation formula is applicable to the function $g(z)$ (see ([1]), p. 137):

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$$g(x+iy)e^{-i\omega}+g(x-iy)e^{i\omega}=2\sum_{-\infty}^{\infty}(-1)^n C_n g\left(x-s+\frac{n\pi}{\sigma+m}\right), \tag{5}$$

where ω is a real parameter,

$$C_n=\frac{\mu y\ \mathrm{Im}\{e^{-i\omega}\sin(s+i\mu y)\}}{(n\pi-s\mu)^2+\mu^2y^2}$$

and s is determined from the equality

$$s\mu=\arg\{\cos(\omega+i\mu y)\},\qquad \mu=\sigma+m.$$

At the same time one can show that

$$|g(u+\mu_n)|\leq$$

$$\leq\varphi(t)\left|\frac{f(u+\mu_n)}{\varphi(u+\mu_n)}\right|\alpha_\varphi(|u+\mu_n-t|)\left[\frac{\sin(u+\mu_n-t)}{u+\mu_n-t}\right]^m\leq M\varphi(t)\left|\frac{f(u+\mu_n)}{\varphi(u+\mu_n)}\right|,$$

where $\mu_n=n\pi/(\sigma+m)$, $u=x-s$.

Owing to this, from (5) we obtain

$$\left|\frac{f(x+iy)e^{-i\omega}+f(x-iy)e^{i\omega}}{\varphi(x)}\right|\cdot\left|\frac{\sin iy}{iy}\right|^m\leq 2M\sum_{-\infty}^{\infty}|C_n|\left|\frac{f(x-s+\mu_n)}{\varphi(x-s+\mu_n)}\right|. \tag{6}$$

Thus, taking into account that

$$\sum_{-\infty}^{\infty}|C_n|=\left[\mathrm{ch}^2(\sigma+m)y-\sin^2\omega\right]^{1/2},$$

the assertion follows from inequality (6):

Theorem 1. For an entire function $f(z)\in W_\sigma^{(p,\varphi)}$, the inequality

$$\left\|f(x+iy)e^{-i\omega}+f(x-iy)e^{i\omega}\right\|_{p,\varphi}\leq$$

$$\leq 2M\left(\frac{y}{\mathrm{sh}\,y}\right)^m\left[\mathrm{ch}^2(\sigma+m)y-\sin^2\omega\right]^{1/2}\|f\|_{p,\varphi}. \tag{7}$$

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2. Let us note that inequality (7) remains valid also for a function $f(z)$ from the class $B_\sigma^{(p,\varphi)}$; it can be written in the form

$$\int_{-\infty}^{\infty} \left|\frac{f(x+iy)}{\varphi(x)}\right|^p \left|1+\lambda(x,y)e^{2i\omega}\right|^p dx \leq$$

$$\leq 2^p M^p \left(\frac{y}{\operatorname{sh} y}\right)^{mp} \left[\operatorname{ch}^2(\sigma+m)y - \sin^3\omega\right]^{p/2} \|f\|_{p,\varphi},$$

where

$$\lambda(x,y) = \frac{f(x-iy)}{f(x+iy)} = \rho(x,y)e^{i\theta}, \qquad 0<\rho(x,y)\leq 1.$$

Integrating both sides of the last inequality with respect to ω from 0 to 2π and observing that

$$\int_0^{2\pi} \left|1+\rho(x,y)e^{i(\theta+2\omega)}\right|^p d\omega \geq 2^{p+1}B\left(\frac{1}{2}p+\frac{1}{2},\frac{1}{2}\right),$$

after slight transformations we arrive at the following conclusion:

Theorem 2. For an entire function $f(z) \in B_\sigma^{(p,\varphi)}$ the inequality

$$\|f(x+iy)\|_{p,\varphi} \leq MD_p[(\sigma+m)y]\operatorname{ch}(\sigma+m)y\left(\frac{y}{\operatorname{sh} y}\right)^m \|f\|_{p,\varphi}, \tag{8}$$

holds, where

$$D_p(u) = \left\{\frac{1}{2B\left(\frac{1}{2}p+\frac{1}{2},\frac{1}{2}\right)}\int_0^{2\pi}\left(1-\sin^2\omega\,\operatorname{sech}^2 u\right)^{p/2} d\omega\right\}^{1/p}, \tag{9}$$

and $B(\alpha,\gamma)$ is Euler' s beta function.

3. For an entire function $f(z) \in B_\sigma^{(p,\varphi)}$, consider again the auxiliary function $g(z)$ defined by equality (2), and note that it satisfies the condition $|g(x+iy)| \leq |g(x-iy)|$; as inequality (4) shows, $g(z)$ belongs to the class $W_{\sigma+m}^{(p)}$. In paper (4) it is proved that for $g(z) \in W_{\sigma+m}^{(p)}$ the inequality

$$\sup_{-\infty<x<\infty} |g(x+iy)| \leq [\omega_p(\sigma+m,y)]^{1/p}\|g\|^p, \tag{10}$$

holds, where

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$$\omega_p(\mu, y) = \begin{cases} \dfrac{\operatorname{sh} p\mu y}{\pi p y}, & \text{for } 1 \le p \le 2, \\ \dfrac{\operatorname{sh} p\mu y}{\pi y}, & \text{for } p > 2. \end{cases}$$

We note that for entire functions $f(z)$ from the class $B_\sigma^{(p)}$, the expression $\omega_p(\mu, y)$ was refined by D. I. Mamedkhanov; namely, it was found that

$$\omega_p(\mu, y) = \frac{s\mu}{\pi} \left[D_{p/s}(s\mu y) \operatorname{ch} s\mu y\right]^{p/s},$$

where $\mu = \sigma + m$ and $D_p(u)$ is defined by equality (9).

From inequality (10), taking into account inequality (4) and then putting $t = x$, we obtain:

$$\left|\frac{f(x+iy)}{\varphi(x)}\right| \le M \left(\frac{y}{\operatorname{sh} y}\right)^m \left(\frac{s\mu}{\pi}\right)^{1/p} \left[D_{p/s}(s\mu y) \operatorname{ch} s\mu y\right]^{1/s}. \qquad (11)$$

Thus, from the inequality

$$\|f(x+iy)\|_{p',\varphi} \le \left\{\sup_{-\infty<x<\infty} \left|\frac{f(x+iy)}{\varphi(x)}\right|\right\}^{(p'-p)/p'} \|f(x+iy)\|_{p,\varphi}^{p/p'}$$

for $1 \le p < p' \le \infty$, by virtue of inequalities (8) and (11), we arrive at the following conclusion:

Theorem 3. For an entire function $f(z) \in B_\sigma^{(p,\varphi)}$, with $1 \le p < p' \le \infty$, the inequality

$$\|f(x+iy)\|_{p',\varphi} \le M \left(\frac{s\mu}{\pi}\right)^{1/p-1/p'} \left(\frac{y}{\operatorname{sh} y}\right)^m \left[D_{p/s}(s\mu y) \operatorname{ch} s\mu y\right]^{(p'-p)/sp'} \left[D_p(\mu y) \operatorname{ch} \mu y\right]^{p/p'} \|f\|_{p,\varphi},$$

holds, where $\mu = \sigma + m$ and $D_p(u)$ is defined by equality (9).

Finally, let us denote by $B_{\sigma_1,\dots,\sigma_n}^{(p,\varphi)}$ the class of entire functions $f(z_1, \dots, z_n)$ of finite degree $\sigma_1, \dots, \sigma_n$ satisfying the conditions:

1°.
$$|f(x_1 + iy_1, \dots, x_n + iy_n)| \le |f(x_1 - iy_1, \dots, x_n - iy_n)|,$$

2°.
$$\|f\|_{p,\varphi}^{(n)} = \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left|\frac{f(x_1, \dots, x_n)}{\varphi(x_1, \dots, x_n)}\right|^p dx_1 \cdots dx_n\right)^{1/p} < +\infty.$$

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Here it is assumed that

$$a_k(t) = \sup_{\substack{-\infty < x_1, \dots, x_n < \infty \\ |y| \le t}} \frac{\varphi(x_1, \dots, x_{k-1}, x_k + y, x_{k+1}, \dots, x_n)}{\varphi(x_1, x_2, \dots, x_n)} \le \sum_{j=0}^{m} A_j t^j$$

$$(k = 1, 2, \dots, n).$$

Theorem 4. For an entire function $f(z_1, \dots, z_n)$ from the class $B^{(p,\varphi)}_{\sigma_1,\dots,\sigma_n}$, the following inequalities hold:

$$1^\circ. \quad \|f(x_1 + iy_1, \dots, x_n + iy_n)\|^{(n)}_{p,\varphi} \le M^n \prod_{k=1}^{n} \Omega_k(y_k) \, \|f\|^{(n)}_{p,\varphi}.$$

$$2^\circ. \quad \|f(x_1 + iy_1, \dots, x_n + iy_n)\|^{(n)}_{p',\varphi} \le M^n \prod_{k=1}^{n} \mathfrak{M}_k(y_k) \, \|f\|^{(n)}_{p,\varphi}$$

for $1 \le p < p' \le \infty$, where $\mu_k = \sigma_k + m$,

$$\Omega_k(y_k) = D_p(\mu_k y_k) \operatorname{ch} \mu_k y_k \left(\frac{y_k}{\operatorname{sh} y_k} \right)^m,$$

$$\mathfrak{M}_k(y_k) =$$

$$= \left(\frac{s\mu_k}{\pi} \right)^{1/p - 1/p'} \left(\frac{y_k}{\operatorname{sh} y_k} \right)^m \left[D_{p/s}(s\mu_k y_k) \operatorname{ch} s\mu_k y_k \right]^{(p'-p)/sp'} \left[D_p(\mu_k y_k) \operatorname{ch} \mu_k y_k \right]^{p/p'},$$

is determined by equality (9).

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