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## Abstract

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# MATHEMATICS

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# THE DIRICHLET PROBLEM FOR AN ELLIPTIC SYSTEM OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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**1.** We consider the Dirichlet problem in a bounded multiply connected domain $D$ with boundary $\Gamma = \Gamma_0 + \Gamma_1 + \dots + \Gamma_m$ for the elliptic system of differential equations

$$L_{xy}(u) \equiv \sum_{i+j=2} A_{ij} \frac{\partial^{i+j} u}{\partial x^i \partial y^j} = h, \tag{1}$$

where $u = (u_1, \dots, u_n)$ is the vector sought; $h = (h_1, \dots, h_n)$ is a given real vector; $A_{ij}$ are square matrices of order $n$, whose elements are real functions of $x, y$. It is assumed that the matrices $A_{ij}$ belong to the class $C_\alpha^{i+j}(\overline{D})$, and $h$ belongs to the class $C_\alpha(\overline{D})$.

For system (1) the Dirichlet problem is posed as follows: it is required to find a solution of system (1), regular in the domain $D$, belonging to the class $C_\alpha^1(\overline{D})$ and satisfying the boundary condition

$$u|_\Gamma = 0. \tag{2}$$

For a sufficiently broad class of elliptic systems, problem (1)–(2) and more general boundary-value problems have been studied in papers ($^{1-4}$). The problems considered in these papers are reduced to an equivalent integral equation of normal type; the Noether property of these problems is proved and the index is computed. In the paper of Ya. B. Lopatinskii ($^5$), a condition on the coefficients of system (1) was obtained which is a sufficient condition for the Noether property of problem (1)–(2). Elliptic systems satisfying this condition form a broader class than the systems considered in papers ($^{1-4}$). For a simply connected domain, problem (1)–(2) was considered in ($^6$). In that paper, problem (1)–(2) is not reduced to an equivalent integral equation; however, under the fulfillment of the indicated condition, the Noether property of this problem is proved and a formula for the index is obtained.

In the present paper a new method is indicated for reducing problem (1)–(2) to an equivalent integral equation, which turns out to be an equation of normal type in all cases when the condition of Ya. B. Lopatinskii is fulfilled. It is proved that the Dirichlet problem for equation (1) and for the adjoint equation form a Noether pair of boundary-value problems if this condition is fulfilled.

The adjoint system to (1) is the system of equations

$$L^*_{xy}(u) \equiv \sum_{i+j\le 2} (-1)^{i+j} \frac{\partial^{i+j}(uA_{ij})}{\partial x^i \partial y^j} = 0. \tag{3}$$

**2.** Let

$$K(x,y,\lambda) = \sum_{i+j=2} A_{ij}\lambda^j \equiv \|a_{ij}(x,y,\lambda)\|,$$

$$\Delta(x,y,\lambda) = \|\Delta_{ij}\|(\det K(x,y,\lambda))^{-1},$$

where $\Delta_{ij}$ $(i,j = 1,2,\dots,n)$ is the algebraic complement of the element $a_{ij}$ in $\det K(x,y,\lambda)$.

Consider the matrix

$$v(x,y,\xi,\eta) = \frac{1}{2\pi^2}\operatorname{Re}\int_\gamma \Delta(x,y,\lambda)\ln(\xi - x + \lambda(\eta - y))\,d\lambda, \tag{4}$$

where $\gamma$ is a contour in the half-plane $\operatorname{Im}\lambda > 0$, enclosing all roots $\lambda$ of the polynomial $\det K(x,y,\lambda)$ that lie in this half-plane.

The elements of the matrix (4) are single-valued functions with respect to $x,y,\xi,\eta$ $((x,y)\ne(\xi,\eta))$. Expression (4) is a Levi function for the system (1).

Transforming the expression

$$\iint_D \Big(WL_{\xi\eta}(u) - L^*_{\xi\eta}(W)u\Big)\,d\xi\,d\eta \tag{5}$$

by the well-known Green formula, assuming that $u(\xi,\eta)$ is a solution of problem (1)-(2) and $W = v(x,y,\xi,\eta)$, where $(x,y)$ is a fixed point in the domain $D$, and $(\xi,\eta)$ is the point of integration, we shall have

$$u(x,y) + \iint_D L^*_{\xi\eta}(v)u(\xi,\eta)\,d\xi\,d\eta = \iint_D vh(\xi,\eta)\,d\xi\,d\eta + \int_\Gamma \Omega\frac{\partial u}{\partial\nu}\,ds, \tag{6}$$

where $\partial/\partial\nu$ is differentiation along the inner normal to $\Gamma$, and

$$\Omega = -v\left(A_{20}(\xi,\eta)\cos^2(\nu,\xi) + A_{11}(\xi,\eta)\cos(\nu,\xi)\cos(\nu,\eta) + A_{22}(\xi,\eta)\cos^2(\nu,\eta)\right).$$

We note that $L^*_{\xi\eta}(v)$ at the point $(\xi,\eta) = (x,y)$ has a pole of order no higher than the first. We represent the matrix $L^*_{\xi\eta}(v)$ in the form of a sum of quadratic matrices of order $n$ as follows:

$$L^*_{\xi\eta}(v) = \|\alpha_{ij}(x,y,\xi,\eta)\| + \sum_{k=1}^{l} \|\beta^{(k)}_{ij}(\xi,\eta)\|\gamma_k(x,y), \tag{7}$$

where

$$\max_{1\le i\le n} \iint_D \sum_{j=1}^{n} |\alpha_{ij}(x,y,\xi,\eta)|\, d\xi\, d\eta < q,$$

$q = \text{const} < 1$, and $\gamma_1(x,y), \dots, \gamma_l(x,y)$ are linearly independent functions in the domain $D$.

The identity (6) can be given the form

$$u = \iint_D M_1 h(\xi,\eta)\, d\xi\, d\eta + \int_\Gamma \Omega_1 \frac{\partial u}{\partial \nu}\, ds + \sum_{k=1}^{l} N_k(x,y) c_k, \tag{8}$$

where

$$M_1 = M(v(x,y,\xi,\eta)), \qquad \Omega_1 = M(\Omega(x,y,\xi,\eta)),$$

$$N_k = M(\gamma_k(x,y)E); \tag{9}$$

$$M(\psi(x,y,\xi,\eta)) = \psi(x,y,\xi,\eta) + \iint_D \widetilde{K}(x,y,t,\tau)\psi(t,\tau,\xi,\eta)\, d\xi\, d\eta; \tag{10}$$

$$c_k = (c_{k1}, \dots, c_{kn}), \qquad c_{ki} = -\iint_D \left(\sum_{j=1}^{n} \beta^{(k)}_{ij}(\xi,\eta) u_j(\xi,\eta)\right) d\xi\, d\eta; \tag{11}$$

$\widetilde{K}(x,y,\xi,\eta)$ is the resolvent of the Fredholm equation with kernel $\|\alpha_{ij}(x,y,\xi,\eta)\|E$, and $E$ is the identity $n$-dimensional matrix.

On the basis of formula (8), the solution of the Dirichlet problem for the system (1) is naturally sought in the form

$$u(x,y) = \iint_D M_1 g(\xi,\eta)\, d\xi\, d\eta + \int_\Gamma \Omega_1 f(s)\, ds + \sum_{i=1}^{l} N_i(x,y) c_i, \tag{12}$$

where $g(\xi,\eta)$ and $f(s)$ are $n$-dimensional vector-functions of class $C_\alpha$, respectively in the domain $D$ and on the contour $\Gamma$; $c_i$ are constant $n$-dimensional vectors.

Substituting (12) into equation (1), we obtain:

$$g(x,y) + \iint_D L_{x,y}(M_1)\, g(\xi,\eta)\, d\xi\, d\eta =$$

$$= -\int_\Gamma L_{xy}(\Omega_1)\, f(s)\, ds - \sum_{i=1}^{k} L_{xy}(N_i(x,y))\, c_i + h(x,y). \tag{13}$$

The kernels $L_{xy}(M_1)$ and $L_{xy}(\Omega_1)$ may have a polar singularity of the first order at the points $(\xi,\eta) = (x,y)$.

Represent the matrix $L_{xy}(M_i)$ in the form (7):

$$L_{xy}(M_1) = \left\|\tilde{a}_{ij}(x,y,\xi,\eta)\right\| + \sum_{k=1}^{l_1} \left\|\tilde{\beta}_{ij}^{(k)}(\xi,\eta)\right\| \gamma_k(x,y)$$

and denote by $\widetilde{M}(x,y,\xi,\eta)$ the resolvent of the Fredholm equation with kernel $\left\|\tilde{a}_{ij}(x,y,\xi,\eta)\right\|$. Then from (13) we obtain (see (7), p. 33)

$$g(x,y) + \sum_{k=1}^{l_1} \widetilde{N}_k(x,y) d_k = H(x,y) + \iint_D \widetilde{M}(x,y,\xi,\eta) H(\xi,\eta)\, d\xi\, d\eta, \tag{14}$$

where $H(x,y)$ is the right-hand side of equation (13);

$$\widetilde{N}_k(x,y) \equiv \left\|b_{ij}^{(k)}\right\| = E\gamma_k(x,y) + \iint_D \widetilde{M}(x,y,\xi,\eta)\gamma_k(\xi,\eta)\, d\xi\, d\eta,$$

$d_k = (d_{k1}, \dots, d_{kn})$ are constant vectors satisfying the system of equations

$$d_{ki} + \sum_{\alpha,j=1}^{n} \sum_{l=1}^{l_1} \left( \iint_D \tilde{\beta}_{ij}^{(k)}(x,y) b_{i\alpha}^{(l)}(x,y)\, dx\, dy \right) d_{l\alpha}$$

$$= -\sum_{j=1}^{n} \iint_D \tilde{\beta}_{ij}^{(k)}(x,y) \delta_j(x,y)\, dx\, dy \tag{15}$$

$$(i = 1, 2, \dots, n; \quad k = 1, 2, \dots, l_1);$$

$\delta = (\delta_1, \dots, \delta_n)$ is the right-hand side of equation (14).

Substituting the value of $g(x, y)$ from (14) into (12), we obtain

$$u(x, y) = \iint_D K_3 h(\xi, \eta)\, d\xi\, d\eta + \int_\Gamma \Omega_{1f}(s)\, ds + \int_\Gamma \Omega_{2f}(s)\, ds +$$

$$+ \sum_{k=1}^{l_1} G_k(x, y) d_k + \sum_{k=1}^{l} \overline{N}_i(x, y) c_i. \tag{16}$$

The matrices $K_3(x, y, \xi, \eta)$, $\Omega_2(x, y, \xi, \eta)$, $G_k(x, y)$, and $\overline{N}_k(x, y)$ are expressed explicitly through $M_1, \widetilde{M}, N_k, \widetilde{N}_k, \Omega_1$. We shall not write them out here; let us only note that the matrix $K_3$ and the first derivatives of the matrix $\Omega_2$ may have a logarithmic singularity at the points $(x, y) = (\xi, \eta)$.

It is clear that the boundary condition (2) is equivalent to the conditions

$$\frac{\partial u}{\partial s} \equiv \frac{\partial u}{\partial x} \cos(x, s) + \frac{\partial u}{\partial y} \cos(y, s) = 0 \quad \text{on } \Gamma; \tag{17}$$

$$u(x_i, y_i) = 0 \quad (i = 0, 1, \dots, m), \tag{18}$$

where $(x_i, y_i)$ are fixed points on $\Gamma_i$ $(\Gamma = \Gamma_0 + \Gamma_1 + \dots + \Gamma_m)$. In what follows we shall assume that in (18), instead of $u(x, y)$, the expression (16) has been substituted.

Suppose that the condition of Ya. B. Lopatinskii(5) is satisfied,

$$\det \left\{ \int_\gamma \Delta(x, y, \lambda)\, d\lambda \right\} \neq 0 \quad \text{on } \Gamma, \tag{19}$$

where $\gamma$ is a contour in the half-plane $\operatorname{Im} \lambda > 0$ enclosing all roots $\lambda$ of the polynomial $\det K(x, y, \lambda)$ that lie in this half-plane.

Then, substituting expression (16) into the boundary condition (17) and passing to the limit as the point $(x, y)$ tends to the boundary of the domain $D$, we obtain a one-dimensional singular integral equation of normal type for determining the vector-function $f$, which, together with conditions (15) and (18), is equivalent to our problem. In particular, from this integral equation and from conditions (15) and (18), by virtue of the known theorems of the theory of singular integral equations (see (8)), it follows that

**Theorem 1.** *If condition (19) is fulfilled, the homogeneous problem (1)–(2) has a finite number of linearly independent solutions, and for solvability of the nonhomogeneous problem (1)–(2) it is necessary and sufficient that the function* $h(x,y)$ *satisfy a finite number of conditions of the form*

$$\iint_D h(x,y)\varphi_i(x,y)\,dx\,dy = 0 \quad (i = 1, 2, \dots, k'), \tag{20}$$

*where* $\varphi_i(x,y) \in C^2$ *in* $D$ *and* $\varphi_i(x,y) \in C^1_\alpha$ *in* $\bar{D}$*.*

Transforming expression (5) by Green' s formula ($W = \varphi_i(\xi,\eta)$), we conclude that the functions $\varphi_i(x,y)$ $(i = 1, 2, \dots, k')$ are solutions of problem (2)–(3), and every solution of problem (2)–(3) satisfies condition (20).

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*Note: Figure translations are in progress. See original paper for figures.*