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Abstract

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ON FOLIATIONS OF CODIMENSION 1 ON MANIFOLDS

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We shall consider compact C^∞-smooth oriented manifolds, closed or with boundary. As usual, we shall say that on a manifold M^n a nonsingular foliation of dimension $k \leq n$, or of codimension $n-k$, is given if on the manifold there is given an integrable field of class C^∞ of tangent k-planes.* In the case where the manifold has boundary $V^{n-1} = \partial M^n$, we require in addition that on the boundary this field of k-planes lie in the $(n-1)$-planes tangent to the boundary. We shall be interested in the case where $k = n-1 > 1$. Consider the field of one-dimensional directions on the manifold M^n, normal to the foliation at each point (it is assumed that some Riemannian metric is given on the manifold M^n).

Definitions. A foliation of codimension 1 is called orientable if there exists a continuous field of vectors of length 1, normal to the foliation at each point (and not merely a field of one-dimensional directions). An **orientation of a foliation** is the choice of such a vector field (there are exactly two such fields, see (^{4})). A foliation with a fixed orientation will be called **oriented**. A **transversal segment**, as usual, is a smooth regular curve in the manifold, nowhere tangent to the foliation and having two distinct ends. In an oriented foliation, on every **transversal segment** there naturally arise directions—positive and negative. The ends of a transversal segment (there are exactly two of them) are divided into the first and the second; namely, we consider it to go from the first end to the second in the positive direction. By a **closed transversal** we shall mean a smooth closed regular curve, at no point tangent to the foliation. On a closed transversal there also arises a positive direction in an oriented foliation.

Haefliger proved the remarkable fact that for $n > 2$ every closed transversal of a foliation of codimension 1 represents an element of the group $\pi_1(M^n)$ of infinite order in the case where the foliation is analytic (^{4}). In particular, on S^3 there are no analytic foliations of codimension 1, although there are infinitely differentiable ones (Reeb' s example (^{4,5})). As usual, a leaf of a foliation is a connected "integral hypersurface," tangent to the foliation at each of its points. The leaf passing through a point $x \in M^n$ will be denoted by A_x.

We shall write $A_x \geq A_y$, where $x, y \in M^n$, if there exists a transversal segment leading from the point x to the point y—in the positive direction. We assume that $A_x \geq A_x$. In view of the connectedness of the leaves A_x and A_y, it is easy to show that for any other pair of points $x_1 \in A_x$, $y_1 \in A_y$ there exists a transversal segment leading from x_1 to y_1 in the positive direction. If $A_x \geq A_y$ and $A_y \geq A_x$, then we shall say that the leaves A_x and A_y belong to one connected component of the foliation. Thus, by a **connected component of the foliation** we mean a set of points on all leaves $\{A_x\}$, for any pair of which A_{x_1} and A_{x_2} we have $A_{x_1} \geq A_{x_2}$, $A_{x_2} \geq A_{x_1}$.

The entire foliation on the manifold M^n decomposes into a union of connected components. Since from $A_x \geq A_y \geq A_z$ it follows that $A_x \geq A_z$, and $A_x \geq A_x$, these components are pairwise disjoint and together give the whole manifold M^n.

* The corresponding French term is *variété feuilleté*.

Theorem 1. Only the following cases are possible:

1. A connected component is a closed connected submanifold $W^{n-1} \subset M^n$, coinciding entirely with a single leaf.
2. The closure of a connected component containing more than one leaf is a submanifold $V^n \subset M^n$ with boundary $\partial V^n = W^{n-1}$, where $W^{n-1} = W_1^{n-1} \cup \cdots \cup W_k^{n-1}$. All parts of the boundary W_i^{n-1} are connected components, and the original connected component itself is the open manifold $V^n \setminus W^{n-1}$; all groups $\pi_1(W_i^{n-1})$ are infinite. Moreover, the component $V^n \setminus W^{n-1}$ has no compact leaves near the boundary.
3. All of the manifold M^n is a single connected component.

The proof of Theorem 1 is not difficult.

Thus, three types of connected components are possible. It is easy to show that if there is a connected closed set of components of the first type, then all these components are diffeomorphic to one another, and this whole connected set is the direct product $W^{n-1} \times I$, where W^{n-1} is one component and I is a closed interval. Components of the second type fill entire pieces of the manifold M^n and are separated by components of the first type, isolated or in families of the form $W^{n-1} \times I$.

Since the relation $\geqslant$ had previously been introduced between leaves, it is now transferred to the connected components, and for the components it is a partial-order relation. Thus, with a foliation η on a manifold M^n (a foliation of codimension 1) we have associated a comparatively simple invariant—the partially ordered set $S(\eta)$ of its components.

For example, for the foliation η constructed by Reeb on S^3 and having only one compact leaf—the torus $T^2 \subset S^3$—the set $S(\eta)$ consists of three elements: $\alpha_1, \alpha_2, \alpha_3 \in S(\eta)$, where $\alpha_1 > \alpha_2 > \alpha_3$.

Obviously, if a foliation on a closed manifold M^n has a leaf everywhere densely filling the whole manifold M^n, then this foliation is connected. For example, such foliations can be constructed on tori or on manifolds of line elements to surfaces of negative curvature, starting from the geodesic flows on these surfaces and their properties (Anosov, Arnold, Sinai) (^{1}). A foliation that is a skew product over the circle is also connected.

Let us now consider one connected component L^n of a foliation on a manifold M^n, containing more than one leaf, and mark a point $x \in L^n$ on some leaf. Consider all transversal circles, positively oriented and passing through the point x. It is easy to define the product ab of such oriented transversals a, b, likewise passing through the point x. We now combine into classes all transversals regularly homotopic to one another, where the regular homotopy is in the class of these same transversals. This regular homotopy, as usual, will be called **connected at the point** x. The product ab is correctly defined on the classes of connected regular homotopy of transversals, and after passing to classes we obtain a semigroup $t(x)$ of classes of connected regular homotopy of transversals at the point $x \in L^n$.

It is easy to show that if two points $x, y \in L^n$ lie on the same leaf $A \subset L^n$, then the semigroups $t(x)$ and $t(y)$ are isomorphic, and this isomorphism $t(x) \to t(y)$ depends only on the path leading from the point x to the point y along the leaf A; moreover, the isomorphism depends on the homotopy class of this path on the leaf A, with the endpoints fixed during the homotopy. Thus, we may denote the semigroup $t(x)$ by $t(A)$, where $x \in A$, and the group $\pi_1(A)$ acts as a group of automorphisms of the semigroup $t(A)$. The orbits of the group $\pi_1(A)$ on $t(A)$ are the classes of free regular homotopy of closed transversals passing through the leaf A; we denote the set of orbits by t_A. One can define a homomorphism

$$\pi_A : t(A) \to \pi_1(L^n) \to \pi_1(M^n),$$

under which the action of the group $\pi_1(A)$ on $t(A)$ is transformed, naturally, into inner automorphisms of the groups $\pi_1(L^n)$, $\pi_1(M^n)$.

Let us indicate several facts that are easily proved:

1. If inside one connected component $L^n \subset M^n$ there lies a compact leaf $A \subset L^n$, then this compact leaf is a basic cycle of the group $H_{n-1}(M^n)$, since there is a closed transversal intersecting it at one point (^{4}).
2. If $n = 3$, and in the manifold M^n there is a compact leaf $A \subset M^3$, homologous to zero in M^3, then this leaf is a torus T^2 (^{2,3}).
3. For the Reeb foliation in the filled torus $D^2 \times S^1$, the semigroup $t(x)$ for an interior point $x \in D^2 \times S^1$ is isomorphic to the semigroup of positive numbers.

We shall now study one-dimensional closed paths lying entirely on a single leaf. Let A be a leaf and $x \in A$. Consider the group $\pi_1(A, x)$.

Introduce the following notions:

a) An element $q \in \pi_1(A, x)$ is called a **right limit cycle** if there is a representative $f : S^1 \to A$, $f(0) = x \in A$, such that on the plaque normal to the leaf A and reconstructed from the curve $f(S^1) \subset A$ in the positive direction, there is a curve, arbitrarily close to the curve $f(S^1)$, and which is the intersection of this plaque with a sufficiently close leaf, which does not close up when the curve $f(S^1)$ is traversed from the point $f(0) = x$ to the point $f(2\pi) = x$.

Similarly one defines the notion of a **left limit cycle**–the plaque is reconstructed in the negative direction of the normal.

b) If two representatives $f, g : S^1 \to A$ are homotopic, then they simultaneously possess these properties, and the definition of a right (left) limit cycle as an element of the group $\pi_1(A, x)$ is correct (see ([4])).

c) The set of elements N_1 $(N_2) \subset \pi_1(A, x)$ which are not right (left) limit cycles is a subgroup and even a normal divisor in $\pi_1(A, x)$. The **groups of limit cycles** will be called the quotient groups $P_i(A) = \pi_1(A)/N_i$, $i = 1, 2$. It is easy to see that the groups $P_i(A)$ have no torsion. Moreover, for oriented foliations the notions of a right and a left cycle are well defined.

Let a, b be two closed transversals on the manifold M^n, such that $A_a \cap A_b = \emptyset$*. The following lemma holds.

Lemma 1. a) *If the transversals a, b are homotopic, then on the manifold M^n there is at least one limit cycle on some leaf; if both transversals a, b lie inside one component $L^n \subset M^n$, and the homotopy lies in L^n, then the limit cycle will also lie on a leaf inside L^n. One may assume that one of the transversals is empty and the other is homotopic to zero.*

b) *On the boundary of a connected component of the foliation there is always a limit cycle on the side from which the whole component adjoins its boundary.*

Let us now consider a smooth closed regular curve l on a leaf $A \subset M^n$, homotopic to zero in the manifold M^n and not homotopic to zero on its leaf $A \subset M^n$. Span the curve by a regular disk $D \subset M^n$, in general position with the leaves. The intersections of the disk D with the leaves are smooth curves with isolated singular points.

We shall call the disk D canonical if it has the following properties:

1. The only singular point inside D is the center, and all intersections of the disk D with the leaves are smooth closed curves.
2. All these curves, except the boundary $\partial D = l$, are homotopic to zero on their leaves by means of regular films.
3. If a leaf B intersects the disk D along a curve lying inside the disk D, then $\pi_2(B) = 0$.

Suppose that the foliation is such that for every leaf $A \subset M^n$ the group $\pi_2(A)$ is trivial (condition 3 is automatically satisfied). There is an important

* By A_a is meant the union of leaves intersecting the transversal a.

Lemma 2. Let, for some leaf $A \subset M^n$, there be an element $a \in t(A)$ such that $\pi_A(a) = 1$, where $\pi_A : t(A) \to \pi_1(M^n)$. Then the foliation contains at least one canonical disk.

Let now D be a canonical disk of the foliation and $l = \partial D$. The leaf passing through l will be denoted by A. For $n = 3$ the following main result holds.

Lemma 3. The leaf A passing through the boundary $l = \partial D$ is a two-dimensional torus T^2 separating the foliation, and one closed interior region is homeomorphic to $D^2 \times S^1$, on which the foliation is homeomorphic to the Reeb foliation. This interior region is one connected component.

Thus we obtain the following assertion:

Theorem 2. Let $n = 3$ and let the group $\pi_1(M^3)$ be finite. Then inside the manifold M^3 there is a compact leaf $T^2 \subset M^3$, separating the foliation (and itself being a connected component), and the closure of one of these connected components is homeomorphic to the Reeb foliation on the solid torus $D^2 \times S^1$.

We shall make several additions to Theorem 2.

1. It is easy to show that for any vector field transverse to the Reeb foliation on $D^2 \times S^1$ there exists a periodic integral trajectory. This follows from the fact that a mapping of the closed disk D^2 into itself has a fixed point. From Theorem 3 we obtain

Corollary. A vector field transverse to a 2-foliation on any three-dimensional manifold M^3 with finite fundamental group always has a periodic integral trajectory.

2. Let $M^3 = S^3$. It is easy to show that if the periodic solution obtained is unknotted in S^3, then there exists another "Reeb component" in the foliation and one more periodic solution of the transverse vector field. Moreover, if there are exactly k Reeb components, then the system of k corresponding solutions of the transverse vector field must be linked, like a system of k circles in S^3, and cannot split into two subsystems one of which is unknotted (these two subsystems cannot be separated from one another by an isotopy in S^3). The proof can be given analogously to the proof of Theorem 2.
3. The theorem on the existence of a compact leaf also holds for nonorientable foliations by passing to a double covering that makes the foliation orientable.

We indicate properties of limiting cycles on analytic foliations:

1. We shall say that an element $q > 1$, $q \in P_1(A)$, if the nearby leaves on the right wind onto the cycle q in the positive direction. For an analytic foliation, always either $q > 1$ or $q^{-1} > 1$, if $q \neq 1$ in $P_1(A)$. We shall say that $q_1 > q_2$ if $q_1 q_2^{-1} > 1$. The group $P_1(A)$ is totally ordered.
2. If all cycles in $P_1(A)$ are "rough," i.e. are wound onto with exponential rate, then the group $P_1(A)$ is free abelian. It can always be shown that the quotient group by the commutator $P_1(A)/[P_1(A), P_1(A)]$ is nontrivial if $P_1(A)$ is nontrivial (for an analytic foliation). It follows from this that the group $H_1(A)$ is nontrivial.

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Note: Figure translations are in progress. See original paper for figures.

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