Reports of the Academy of Sciences of the USSR

1964

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1964. Volume 156, No. 2

MATHEMATICS

VERA TRNKOVA

ON THE CLOSURE OF CLASSES OF SPACES UNDER ω-MAPPINGS

(Presented by Academician P. S. Aleksandrov on 8 I 1964)

Definition 1. Let P and Q be topological spaces (only spaces satisfying the Hausdorff separation axiom are considered). Let there be an open cover ω of the space P and a continuous mapping f of the space P into Q. If there exists an open cover α of the space Q such that the system $\{f^{-1}A\}$, $A \in \alpha$, is inscribed in ω, then f is called an ω-**mapping***.

Let us note that if $fP \subseteq Q_1 \subseteq Q$, then an ω-mapping f of the space P into Q is, of course, also an ω-mapping if it is considered as a mapping of P into Q_1. However, in general, an ω-mapping of P into Q_1 may fail to be an ω-mapping if it is considered as a mapping into Q.

Definition 2. Let $\mathcal{X}$ be any class of topological spaces. We shall say that a space Y is **generated by the class $\mathcal{X}$ by means of ω-mappings** (or, more briefly, that Y is ω-**generated by the class** $\mathcal{X}$), if for every open cover ω of the space Y there exists an ω-mapping of the space Y into some X from the class $\mathcal{X}$ (depending, in general, on ω). The class of all spaces ω-generated by the class $\mathcal{X}$ will be denoted by $O(\mathcal{X})$ and called the **closure of the class $\mathcal{X}$ under ω-mappings**; we shall also say that $\mathcal{X}$ ω-**generates the class** $O(\mathcal{X})$. If $\mathcal{X}$ consists of only one space X, we shall write $O(X)$ instead of $O(\mathcal{X})$ and shall speak of the class ω-**generated by the space** X. If $\mathcal{X}$ is a class of spaces and $O(\mathcal{X}) = \mathcal{X}$, then the class $\mathcal{X}$ will be called **closed under ω-mappings**.

It is obvious that the intersection of classes closed under ω-mappings is also closed under ω-mappings. It is easy to see that the following classes are closed under ω-mappings: the class of all Hausdorff spaces, all regular spaces, all completely regular spaces, all normal spaces, all collectionwise normal spaces, all countably paracompact spaces, all paracompact spaces, all strongly paracompact spaces, all (m, n)-compact spaces ([1,3]).

However, for example, the class N of perfectly normal spaces (i.e., normal spaces in which every closed set is a G_δ-set) is not closed under ω-mappings, since, for example, every bicompact space with a single nonisolated point belongs to $O(N)$. It is easy to see that all spaces in N are countably paracompact and normal; on the other hand, one can show that the hereditarily normal space of all countable ordinal numbers does not belong to $O(N)$.

In the following assertions (some of which are obtained directly or are merely a formulation of already known theorems—

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* P. S. Aleksandrov systematically considered ω-mappings of normal spaces (for finite ω) in the paper (^{3b}) of 1947 (for bicompact spaces in (^{3b}) of 1940). The metric analogue (for compacta, the "covering ω-mappings" ("ε-mappings") was already laid by P. S. Aleksandrov in the second half of the 1920s as the basis of his investigations in dimension theory (see especially (^{3a}) and earlier publications, as well as a number of subsequent works by A. N. Tikhonov, K. Kuratowski, S. Eilenberg, K. Daukher, and others).

in terms of ω-generated classes) we shall use the following notation: a segment is denoted by J, a line by E, their n-th powers, $n = 1, 2, \ldots$, respectively by J^n, E^n, and the product of $\aleph_0$ segments (lines) by J^∞ (respectively, E^∞).

Theorem 1. The space J^∞ ω-generates the class of all bicompacts. For $n = 1, 2, \ldots$ the space J^n generates the class of all those bicompacts into each open cover of which one can inscribe an open cover whose nerve is realized by a finite geometric complex in E^n.

Theorem 2. The space E^∞ ω-generates the class of all regular finally compact spaces. For $n = 1, 2, \ldots$ the space E^n ω-generates the class of all those regular spaces into each open cover of which one can inscribe an open cover whose nerve is realized by a star-finite geometric complex in E^n.

Theorem 3. The class of all E^n, $n = 1, 2, \ldots$, ω-generates the class of all those regular spaces into each open cover of which one can inscribe a countable open cover of finite order.

Theorem 4. A discrete space of infinite cardinality $\mathfrak{m}$ ω-generates the class of all zero-dimensional paracompact spaces into each open cover of which one can inscribe an open cover of cardinality $\mathfrak{m}$. The class of all discrete spaces ω-generates the class of all zero-dimensional paracompact spaces.

Theorem 5. If D is a discrete space of infinite cardinality $\mathfrak{m}$, then $D \times E^\infty$ ω-generates the class of all those strongly paracompact spaces into each open cover of which one can inscribe an open cover of cardinality $\leqslant \mathfrak{m}$. The class of all $D \times E^\infty$ with discrete D ω-generates the class of all strongly paracompact spaces.

Theorem 6. If D is a discrete space of infinite cardinality $\mathfrak{m}$, then the class of all $D \times E^n$, $n = 1, 2, \ldots$, ω-generates the class of all those regular spaces into each open cover of which one can inscribe a star-finite open cover of finite order and cardinality $\leqslant \mathfrak{m}$. The class of all $D \times E^n$, where D is discrete, $n = 1, 2, \ldots$,

ω-generates the class of all those regular spaces into each open cover of which one can inscribe a star-finite open cover of finite order.

Theorem 7. The class of all $D \times J^\infty$, where D is discrete, ω-generates the class of all those strongly paracompact spaces all connected components of which are bicompact and possess a complete system of neighborhoods consisting of open-and-closed sets.

Theorem 8. The class $B \times E^\infty$, where B is a generalized Baire space$*$, ω-generates the class of all completely paracompact spaces$**$.

For any nonempty set T denote by $m(T)$ the normed linear space of all bounded real-valued functions on T with norm $\|x\| = \sup_t |x(t)|$.

Theorem 9. The class of all $m(T)$ ω-generates the class of all paracompact spaces.

Remark. This theorem is essentially known; see, for example, ([2]). If $\mathcal{X}$ and $\mathcal{Y}$ are classes of spaces, then we shall denote by $\mathcal{X} \times \mathcal{Y}$ the class of all products $X \times Y$, where $X \in \mathcal{X}$, $Y \in \mathcal{Y}$. It is easy to see that the operator O does not commute with multiplication of spaces and that even the relations

$$O(\mathcal{X} \otimes \mathcal{Y}) \subseteq O(\mathcal{X}) \otimes O(\mathcal{Y}),$$

$$O(\mathcal{X} \otimes \mathcal{Y}) \supseteq O(\mathcal{X}) \otimes O(\mathcal{Y}),$$

* That is, the product of a countable number of discrete spaces.

** A space is called completely paracompact if into each of its open covers one can weakly inscribe a union of a countable number of star-finite covers (a cover β is called weakly inscribed in a cover α if there is a refinement $\gamma \subseteq \beta$ which is inscribed in α in the usual sense).

generally speaking, do not hold. Indeed, let $\mathcal{X}$ be the class of all discrete spaces, and $\mathcal{Y}$ the class of all finally compact spaces. Let P_1 be a metrizable space with a single nonisolated point ξ, which is not locally separable at it. Denote by P_2 the interval $(0, 1)$, and set $P = P_1 \times P_2$; identifying in P all points (ξ, x), $x \in P_2$, we obtain the space P^*. It is easy to see that

$$P \in O(\mathcal{X}) \otimes O(\mathcal{Y}), \qquad P \bar{\in} O(\mathcal{X} \otimes \mathcal{Y}),$$

$$P^* \in O(\mathcal{X} \otimes \mathcal{Y}), \qquad P^* \bar{\in} O(\mathcal{X}) \otimes O(\mathcal{Y}).$$

Theorem 10. *Let the class $\mathcal{X}$ consist of paracompact spaces. Then every space $B \times Y$, where B is bicompact and $Y \in O(\mathcal{X})$, is ω-generated by the class of all $X \times J^\infty$, where $X \in \mathcal{X}$.*

As corollaries we obtain the known theorems asserting that multiplication by a bicompact space preserves paracompactness, strong paracompactness, etc.

Received
30 XII 1963

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Note: Figure translations are in progress. See original paper for figures.