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Abstract

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CLASSIFICATION OF SEMISIMPLE LIE ALGEBRAS OVER A p-ADIC FIELD

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The problem of classifying simple Lie algebras over a field of characteristic 0 has recently been considered by many authors ($^{1-3}$) and others. A. Weil obtained results that make it possible to give a classification of classical Lie groups not containing components of type D_4, over an arbitrary field k of characteristic 0. Analogues of Weil' s theorems were proved for groups of types G_2 and F_4. In the present note a classification is given of all simple Lie algebras over a p-adic field k. The results are obtained by general methods, making use mainly of root techniques.

By a **Lie algebra** we shall mean a Lie algebra over the universal domain. It is defined over a field k if the commutation operation is defined over the field k; in this case the set $\mathfrak{g}_k$ of points rational over k of the algebra $\mathfrak{g}$ is a Lie algebra over k in the classical sense. An algebra defined over a field k is said to be k-**simple** if it contains no ideals defined over k. An algebra is called **absolutely simple** if it is L-simple for every extension L of its field of definition k.

Let $\mathfrak{g}$ be a semisimple Lie algebra defined over a field k. It decomposes over k into a direct sum of k-simple algebras. Therefore, for our purposes it suffices to consider the case where $\mathfrak{g}$ is a k-simple Lie algebra. Then there exists a finite separable extension K of the field k and an absolutely simple algebra $\mathfrak{g}'$, defined over K, such that $\mathfrak{g}$ is obtained from $\mathfrak{g}'$ by restricting the base field from K to k: $\mathfrak{g} = R_{K/k}\mathfrak{g}'$ (^{4}). Thus the problem of classifying semisimple Lie algebras defined over the field k is reduced to the problem of classifying absolutely simple Lie algebras defined over k and its finite extensions. In the classification, Satake' s results (^{3}) were used extensively.

Let k be a p-adic field (a finite extension of the field of p-adic numbers), and let $\mathfrak{g}$ be an absolutely simple Lie algebra defined over k.

Theorem 1. *There exists a maximal subalgebra in $\mathfrak{g}$, consisting of nilpotent elements and defined over an unramified extension $\mathfrak{K}$ of the field k.*

Using Theorem 1 and some results of (^{3}), one can find a normal splitting field K of the algebra $\mathfrak{g}$, containing the field $\mathfrak{K}$, such that: a) if $\mathfrak{g}$ is an algebra of type B_n, C_n, E_7, E_8, G_2, or F_4, then $K = \mathfrak{K}$; b) if $\mathfrak{g}$ is an algebra of type A_n, D_n ($n \neq 4$), or E_6, then the Galois group $\Gamma(K/k)$ is abelian and the group $\Gamma(\mathfrak{K}/k)$

has index 1 or 2 in $\Gamma(K/k)$; c) if $\mathfrak{g}$ is of type D_4, then the group $\Gamma(K/k)$ is either trivial, or isomorphic to one of the cyclic groups Z_2, Z_3, or isomorphic to the symmetric group on three letters S_3.

The formulated results make it possible to give a classification of absolutely simple Lie algebras defined over the field k. By analogy with the real case, we shall call Lie algebras defined over k k-**forms**.

* After the paper had already been written, the author learned that T. A. Springer had proved a more general theorem (^{5}).

Before turning to the description of k-forms, let us introduce some notation and definitions.

If Σ is the root system of a semisimple Lie algebra, and Γ is a subgroup of the group $\operatorname{Aut}\Sigma$, then a Γ**-order** is such a linear order that, if $\alpha \in \Sigma$, $\alpha > 0$, and $\sum_{\sigma\in\Gamma} \sigma\alpha \neq 0$, then $\sigma\alpha > 0$ for all $\sigma \in \Gamma$. The fundamental system of roots with respect to a Γ-order is called a Γ**-fundamental system of roots** (^{3}).

$\mathfrak{g}$	$\Gamma(K/k)$	e	f	Scheme	$\mathfrak{g}$	$\Gamma(K/k)$	e	f	Scheme
A_n	Z_2	$\begin{cases}1\\2\end{cases}$	$\begin{cases}2\\1\end{cases}$	paired A_n-type root scheme, with arrows joining upper and lower nodes; also a folded end scheme	D_{2m}	Z_2	1	2	D-type chain ending in a fork; filled and open nodes as shown

$\mathfrak{g}$	$\Gamma(K/k)$	e	f	Scheme	$\mathfrak{g}$	$\Gamma(K/k)$	e	f	Scheme
A_{2s+1}	Z_2	$\begin{cases}1\\2\end{cases}$	$\begin{cases}2\\1\end{cases}$	paired A-type root scheme with a folded terminal pair	D_{2m+1}	Z_2	$\begin{cases}1\\2\end{cases}$	$\begin{cases}2\\1\end{cases}$	D-type chain ending in a fork; mixed filled/open nodes as shown
A_{ms-1}	Z_m	1	m	linear chain with alternating filled/open segments labeled $k_{m,l}$, $k_{m,l}$, $k_{m,l}$, $k_{m,l}$; below: $0 < l < m/2$, $(l, m) = 1$	D_{2m+1}	Z_4	1	4	D-type chain with terminal vertical pair; label $k_{4,?}$ at the end

$\mathfrak{g}$	$\Gamma(K/k)$	e	f	Scheme	$\mathfrak{g}$	$\Gamma(K/k)$	e	f	Scheme
B_n	Z_2	1	2	B-type chain with arrowed terminal root	D_4	Z_3	$\begin{cases}1\\3\end{cases}$	$\begin{cases}3\\1\end{cases}$	triangular triality D_4 scheme
C_{2m}	Z_2	1	2	C-type chain with double arrow at the left terminal side as shown	D_4	$S_3 =$ Aut Δ	$\begin{cases}3\\6\end{cases}$	$\begin{cases}2\\1\end{cases}$	D_4 triality scheme with six-fold symmetry as shown

$\mathfrak{g}$	$\Gamma(K/k)$	e	f	Scheme	$\mathfrak{g}$	$\Gamma(K/k)$	e	f	Scheme
C_{2m+1}	Z_2	1	2	C-type chain with double arrow at the right terminal side as shown	E_6	Z_2	$\begin{cases}1\\2\end{cases}$	$\begin{cases}2\\1\end{cases}$	E_6-type folded scheme with arcs over the chain and one lower node
D_n	Z_2	$\begin{cases}2\\1\end{cases}$	$\begin{cases}1\\2\end{cases}$	D-type chain ending in a fork with open terminal nodes	E_6	Z_3	1	3	E_6-type scheme labeled $k_{3,1}$ on both sides

D_n	Z_2	1	2	D-type chain ending in a fork with filled terminal nodes	E_7	Z_2	1	2	E_7-type chain with one lower node and mixed filled/open nodes

List of k-forms of absolutely simple Lie algebras (the list does not contain k-forms decomposable over k)

If the algebra $\mathfrak{g}_k$ contains no nilpotent elements, then the algebra $\mathfrak{g}$ is called k**-compact** (^{3}). According to Satake' s theorem (^{3}), a k-form of a semisimple algebra $\mathfrak{g}$ is completely determined by specifying its normal splitting field L, a $\Gamma(L/k)$-fundamental system of roots Δ, the action of the group $\Gamma(L/k)$ on Δ, and the k-compact k-form corresponding to the $\Gamma(L/k)$-subsystem

$$\Delta_0 = \{\alpha \in \Delta : \sum_{\sigma \in \Gamma} \sigma\alpha = 0\}.$$

If $\mathfrak{A}$ is an associative algebra defined over k, $\mathfrak{B}$ is its commutator algebra, and $\mathfrak{Z}$ is the center of the algebra $\mathfrak{B}$, then by $\mathfrak{g}(\mathfrak{A})$ we denote the Lie algebra $\mathfrak{B}/\mathfrak{Z}$. If $\mathfrak{A}_k$ is a division algebra, then $\mathfrak{g}(\mathfrak{A})$ is a k-compact algebra. From our results it follows that the converse assertion is true:

Theorem 2. Let $\mathfrak{g}$ be a k-simple k-compact Lie algebra. There exists a finite extension L of the field k and a division algebra $\mathfrak{D}$ over L such that

$$\mathfrak{g} = R_{L/k}\mathfrak{g}(\mathfrak{D}).$$

It is known that the isomorphism classes of division algebras of order n over k can be numbered by integers m:

$$0 < m < n, \qquad (m, n) = 1.$$

We shall denote the division algebra over k specified by the pair $(n; m)$ by $\mathfrak{D}_{n,m}$. Put further $k_{n,m} = \mathfrak{g}(\mathfrak{D}_{n,m})$; then we have $k_{n,m} = k_{n,n-m}$.

For the description of k-forms we use $\Gamma(K/k)$-fundamental systems of roots (K is the splitting field constructed above) (see the scheme). Roots belonging to Δ_0 are denoted by black circles. The schemes of k-simple k-compact k-forms are singled out, and next to them it is indicated which k-compact k-form they correspond to; to one isolated black circle there always corresponds the algebra $k_{2,1}$. If in the group $\Gamma(K/k)$ there are elements preserving system of simple roots, then the corresponding substitution is described by arrows.

If $\mathfrak{P}$ (respectively $\mathfrak{p}$) is a prime ideal of the field K (respectively k), we put $\mathfrak{p} = \mathfrak{P}^e$, $N_{K/k}(\mathfrak{P}) = \mathfrak{p}^f$. For all k-forms the splitting field K will be specified by giving the numbers e and f and the group $\Gamma(K/k)$. In this case, distinct fields with the same invariants e and f and group $\Gamma(K/k)$ correspond to distinct k-forms, and to every field with the indicated characteristics there corresponds a k-form.

The methods applied to the study of Lie algebras over a p-adic field, together with known results on associative algebras, make it possible to obtain the following propositions (in which we call a Lie algebra defined over k k-quasinormal if it contains a maximal subalgebra defined over k and consisting of nilpotent elements).

Theorem 3. *Let k be an algebraic number field; V the set of its inequivalent valuations; k_v the completion of k with respect to $v \in V$; $\mathfrak{g}$ a Lie algebra defined over k. Then: 1) for almost all $v \in V$ the algebra $\mathfrak{g}$ is k-quasinormal; 2) if for all $v \in V$ the algebra $\mathfrak{g}$ is k_v-quasinormal (respectively split over k_v), then it is k-quasinormal (respectively split over k); 3) there exists a cyclic extension K of the field k such that the algebra $\mathfrak{g}$ is K-quasinormal.*

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CITED LITERATURE

(1) A. Weil, J. Indian Math. Soc., **24**, 289 (1960).
(2) J. Tits, Coll. sur la théorie des groupes algébriques, Bruxelles, 1962, p. 137.
(3) I. Satake, J. Math. Soc. Japan, **15**, No. 2, 210 (1963).
(4) A. Weil, Adeles and Algebraic Groups, Princeton, 1961.
(5) T. A. Springer, Coll. sur la théorie des groupes algébriques, Bruxelles, 1962, p. 129.

Note: Figure translations are in progress. See original paper for figures.

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