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Abstract

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ON THE FORMULATION OF THE SCATTERING PROBLEM AT A FINITE DISTANCE

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In the existing theory of collisions ([1,2]), the states before and after the collision are considered only in the limiting sense, upon removal to infinity. In reality, however, both in applying the solution of this problem in kinetic theories and in experimental studies, one has to fix both the initial conditions and the result of the collision at finite distances. It is assumed that these distances are sufficiently large for the applicability of asymptotic formulas. However, the question of estimates and refinements still remains open.

In the present note, the following terms of the asymptotic expansion of the radial flux are studied, and certain aspects of the formulation of the scattering problem at a finite distance are discussed.

Let us consider stationary elastic scattering by a potential $U = U(r, \vartheta, \varphi)$, which for $r > R$ becomes central. The scattering problem essentially consists in the following: knowing the flux entering the scattering region, to find the flux leaving it. Consequently, it is necessary to be able to separate the incoming and outgoing fluxes from the wave function. However, at a finite distance and for a finite wave number k, an exact separation of the fluxes is impossible, since interference terms will always be present, indicating the interaction of the fluxes. Therefore, at a finite distance the formulation and solution of the wave scattering problem can be only approximate, and the level of accuracy will be determined by the magnitude of the interference terms.

Another feature that arises at a finite distance is connected with the insufficiency of the usual definition of the radial flux in the direction (ϑ, φ) by the expression

$$\frac{r^2}{2ik}\left(\psi^*\frac{\partial\psi}{\partial r} - \psi\frac{\partial\psi^*}{\partial r}\right). \tag{1}$$

The point is that, since the flux is perceived only on an area whose dimensions are not smaller than the wavelength (the uncertainty principle), expression (1) should be averaged over the solid angle $\Delta\omega \sim (kr)^{-2}$.

Let us expand the wave function ψ, satisfying the equation

$$\Delta\psi + (k^2 + U)\psi = 0, \tag{2}$$

in the series

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} X_{lm}(r) Y_{lm}(\vartheta, \varphi) \tag{3}$$

in spherical functions

$$Y_{lm}(\vartheta, \varphi) \sqrt{\frac{(l-|m|)!(2l+1)}{(l+|m|)!\,4\pi}}\, P_l^{|m|}(\cos\vartheta) e^{im\varphi}. \tag{4}$$

For $r > R$ the coefficients of the series (3)

$$X_{lm}(r) = c_{lm}^{+} X_l^{+}(r) - c_{lm}^{-} X_l^{-}(r) \tag{5}$$

are solutions of the radial equations

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dX_l}{dr}\right) + \left(k^2 + U(r) - \frac{l(l+1)}{r^2}\right) X_l = 0. \tag{6}$$

In view of (1), let us subject two linearly independent solutions of equation (6), $X_l^{+}(r)$ and $X_l^{-}(r)$, to the conditions

$$X_l^{\pm} = X_l^{\mp *}, \tag{7}$$

$$X_l^{+} X_l^{-\prime} - X_l^{+\prime} X_l^{-} = \frac{2}{ikr^2}. \tag{8}$$

Then the waves $X_l^{-} Y_{lm}$ turn out to be incoming (the radial flux is negative), while $X_l^{+} Y_{lm}$ are outgoing (the radial flux is positive). Substituting the series (3) into (1), with account of (5) and (7), we obtain the following expression for the total radial flux:

$$\frac{r^2}{2ik} \sum_{l,n=0}^{\infty} \sum_{m=-l}^{l} \sum_{p=-n}^{n} \left\{ c_{lm}^{-*} c_{np}^{-} (X_l^{+} X_n^{-\prime} - X_l^{+\prime} X_n^{-}) + \right.$$

$$+ c_{lm}^{+*} c_{np}^{+} (X_l^{-} X_n^{+\prime} - X_l^{-\prime} X_n^{+}) - c_{lm}^{-*} c_{np}^{+} (X_l^{+} X_n^{+\prime} - X_l^{+\prime} X_n^{+}) +$$

$$\left. + c_{lm}^{+*} c_{np}^{-} (X_l^{-\prime} X_n^{-} - X_l^{-} X_n^{-\prime}) \right\} \langle Y_{lm}^{*} Y_{np} \rangle, \tag{9}$$

where the symbol $\langle ... \rangle$ denotes averaging over $\Delta\omega$. Let us now investigate the asymptotic behavior of the various terms in (9) for large kr. When $l = n$, the expressions in the last two curly brackets (the interference terms) vanish, while in the first two the dependence on kr is isolated by virtue of (8). When $l \neq n$ and l or n exceeds kr, the function $Y_{lm}^* Y_{np}$ has a sinusoidal character with a period smaller than the wavelength, and these terms, roughly speaking, are annulled by the averaging. More definitely about the terms with $l \neq n \gtrless kr$ will be possible to say later. For $l, n < kr$ we shall use the asymptotic expansions of the solutions of equation (6).

For definiteness let us consider the case of a finite potential. Then the functions

$$X_l^{\pm}(r) = \sqrt{\frac{\pi}{2kr}}\, H_{l+1/2}^{(1)}(kr) \tag{10}$$

satisfy conditions (7)–(8), and [3]

$$X_l^{\pm} X_n^{\mp\prime} - X_l^{\pm\prime} X_n^{\mp} = \frac{\pm 2}{ikr^2}\, i^{\mp(l-n)} \left\{ 1 \mp \frac{l(l+1) - n(n+1)}{2ikr} + \right.$$
$$\left. + \frac{[l(l+1) - n(n+1)]^2}{2(2ikr)^2} \mp ... \right\}, \tag{11}$$

$$X_l^{\pm\prime} X_n^{\pm} - X_l^{\pm} X_n^{\pm\prime} = \frac{\pm 2}{ikr^2}\, i^{\mp(l+n)} e^{\pm 2ikr} \left\{ \frac{l(l+1) - n(n+1)}{(2ikr)^2} + ... \right\}. \tag{12}$$

It follows from this that, with a certain accuracy (depending on $c_{lm}^{\pm}$), it is possible to represent the flux (9) in the separated form

$$I^{+}(\vartheta, \varphi; r) - I^{-}(\vartheta, \varphi; r), \tag{13}$$

where

$$I^{\pm}(\vartheta, \varphi; r) = I_0^{\pm}(\vartheta, \varphi; r) \pm \frac{1}{kr} I_1^{\pm}(\vartheta, \varphi; r), \tag{14}$$

$$I_0^{\pm}(\vartheta, \varphi; r) = \langle f_0^{\pm}(\vartheta, \varphi) f_0^{\pm *}(\vartheta, \varphi) \rangle, \tag{15}$$

$$I_1^{\pm}(\vartheta, \varphi; r) = \mathrm{Im} \langle f_0^{\pm}(\vartheta, \varphi) f_1^{\pm *}(\vartheta, \varphi) \rangle, \tag{16}$$

$$f_\nu^{\pm}(\vartheta,\varphi)=\frac{1}{k}\sum_l\sum_{m=-l}^{l}c_{lm}^{\pm}i^{\mp l}[l(l+1)]^{\nu}Y_{lm}(\vartheta,\varphi),\quad \nu=0,1. \tag{17}$$

Thus, at a finite distance the radial fluxes can be determined by expressions (14) (together with (15)–(17)).

In the case of a potential of infinite range, the form of formulas (14)–(17) may change, but in other variants as well one flux will be determined by the amplitudes of the incoming waves c_{lm}^-, and the other by the amplitudes of the outgoing waves c_{lm}^+. A formulation of the problem of finding the coefficients c_{lm}^+ from a given set of coefficients c_{lm}^- was given in ([4]). As $kr\to\infty$, such a formulation of the scattering problem is essentially equivalent to the usual one and is only formally more convenient from the point of view of extending the method of separation of variables to the case of a noncentral potential ([4]).

For finite kr, the separation of the incoming and outgoing fluxes makes possible an approach to the scattering problem that contains a number of new possibilities:

1) The transfer of the element of approximation from the solution to the formulation, bringing the problem closer to the physical conditions and simplifying its mathematical solution by neglecting quantities that have no physical meaning already at the stage of formulating the problem. As the physical interpretation, preliminary analytic estimates, and numerical calculations show, outside the accuracy established by the interference terms there turn out to be, in particular, terms in ([9]) with $l\neq n\gtrless kr$. Owing to this, the problem reduces to finding a finite number of amplitudes c_{lm}^+ from a finite set of amplitudes c_{lm}^-, from a finite system of linear algebraic equations ([4]).

The need for the ability to find certain integral scattering characteristics by direct methods also arises in solving kinetic problems by the method of moments ([5]).

2) Allowance for the interference between the scattered wave and the transmitted part of the incident wave (([2]), p. 127). Let the incident wave in the absence of the scatterer be expanded in the series

$$\psi^0=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(c_{lm}^0X_l^+-c_{lm}^-X_l^{\mp})\,Y_{lm}. \tag{18}$$

The physical result of scattering is usually characterized by the differential cross section, determined from the scattered wave

$$\psi^1=\psi-\psi^0=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(c_{lm}^+-c_{lm}^0)X_l^+Y_{lm} \tag{19}$$

by the formula

$$I^1(\vartheta, \varphi) = \lim_{r\to\infty} \frac{r^2}{2ik} \left(\psi^{1*} \frac{\partial \psi^1}{\partial r} - \psi^1 \frac{\partial \psi^{1*}}{\partial r} \right). \qquad (20)$$

The outgoing flux I^+ contains, in addition to I^1, also contributions from ψ^0 (I^0) and from the interference between ψ^0 and ψ^1 (I^{01}). As $kr \to \infty$, I^+ and I^1 differ (in the case where ψ^0 is a plane wave) only at the point $\vartheta = 0$ (the optical theorem). However, for finite kr the effect of interference between ψ^0 and ψ^1 appears not only at $\vartheta = 0$. Therefore, at small ϑ, differences are possible between I^+ and $I^1 + I^0$ in the spirit of the results of work ([6]).

3) Uniform transition to the classical limit $k \to \infty$. The availability of a prelimit formulation of the problem with finite k and r makes it possible to carry out the limiting transition $k \to \infty$, $r \to \infty$ in different orders. The usual formulation of the scattering problem allows k to increase without bound only after r, as a result of which effects of asymptotic nonuniformity are revealed ([2]). Thus, in the problem of scattering of a plane wave by an ideally hard sphere, for the total scattering cross section in the limit $k \to \infty$ one obtains a result twice as large as the classical value ([2], p. 133). If, however, k is increased faster than r, this paradox does not arise.

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Note: Figure translations are in progress. See original paper for figures.