H. Zieschang

1964

SovietRxiv

Abstract

Full Text

H. Zieschang

ON AUTOMORPHISMS OF PLANAR GROUPS

(Presented by Academician P. S. Aleksandrov on 6 XI 1963)

Let a group $\mathfrak{G}$ of automorphisms preserving orientation* act on a cellular decomposition N of the plane. $\mathfrak{G}$ is called a **planar group**. In what follows we consider only groups with bicompact fundamental domain**, among which are found all fundamental groups of orientable closed surfaces. For them J. Nielsen (^{1}) proved that every automorphism (up to an inner automorphism) is induced by a homeomorphism of the surface. It turns out that this theorem is quite equivalent to a purely algebraic proposition (^{2}), and this connection, in my opinion, is of particular interest.

Here we shall establish the corresponding equivalence for all planar groups, and then prove an algebraic proposition from which follows a geometric theorem generalizing Nielsen' s theorem.

§ 1. Let $\mathfrak{G}$ be a planar group with bicompact fundamental domain. We glue the fundamental domain along equivalent arcs and obtain an orientable closed surface N^*. N covers N^*, generally speaking, with branching. The group $\mathfrak{G}$ may have some fixed points. To each of their equivalence classes there belongs a vertex on N^*—these are the branch points.

The structure of the planar group $\mathfrak{G}$ is the following***:

$$\text{generators} \qquad S_1, \dots, S_m, T_1, U_1, \dots, T_g, U_g; \tag{E}$$

$$\text{defining relations} \quad S_1^{k_1} = \dots = S_m^{k_m} = S_1 \cdot \dots \cdot S_m \prod_{i=1}^{g} [T_i, U_i] = 1.$$

For a group $\mathfrak{G}$ with the given generators and defining relations one can construct a **planar group image**, i.e. a two-dimensional complex K, whose vertices and arcs from a single vertex can be denoted by the elements and generators of the group $\mathfrak{G}$ in such a way that an arc denoted by X leads from the element W to WX. Then the one-dimensional skeleton is the group image in the sense of Dehn, or the Cayley diagram. To every path in the one-dimensional skeleton there corresponds a word in the generators. The relations are obtained from closed paths. If, moreover, the boundaries of the disks of K give the defining relations from (E), then K is called a **planar group image with the given generators and relations**. Since every relation is an expression in the defining relations, K is a planar complex.

It can be proved that every group of structure (E) has a group image that is planar or "spherical."

---

[*] The cells of the decomposition are called, according to their dimension, **vertices**, **arcs**, and **disks**. We assume that there are no fixed disks.

[**] A planar closed subcomplex is called a **fundamental domain** of the group $\mathfrak{G}$ if it is connected and contains exactly one disk from each equivalence class (under automorphisms from $\mathfrak{G}$), together with its entire boundary.

[***] Here $[T_i, U_i]$ denotes the commutator $T_{iU_{iT}}i^{-1}U_i^{-1}$; m is the number of inequivalent fixed points, $k_1, \dots, k_m$ are their orders; g is the genus of the surface N^*; m or g may be zero.

If a vertex of the group image and a word in the generators are given, then there corresponds to them exactly one path on the one-dimensional skeleton which starts from the chosen point.

§ 2. Let $\hat{\mathfrak{G}}$ be the free group with free generators $\hat{S}_1, \dots, \hat{S}_m, \hat{T}_1, \hat{U}_1, \dots, \hat{T}_g, \hat{U}_g$. The standard homomorphism $\hat{\mathfrak{G}} \to \mathfrak{G}$ throws away the hats. An endomorphism of the group $\hat{\mathfrak{G}}$ induces an endomorphism of the group $\mathfrak{G}$, if it maps the kernel of the standard homomorphism into itself.

Theorem 1. *Every automorphism α of the group $\mathfrak{G}$ is induced by an automorphism $\hat{\alpha}$ of the group $\hat{\mathfrak{G}}$, which satisfies the following equations:*

$$\hat{\alpha}\hat{S}_i = \hat{L}_i \hat{S}_{n_i} \hat{L}_i^{-1}, \qquad i = 1, \dots, m;$$

$$\hat{\alpha}(\hat{S}_1 \cdots \hat{S}_m \prod [\hat{T}_i, \hat{U}_i]) = \hat{L} \cdot \hat{S}_1 \cdots \hat{S}_m \prod [\hat{T}_i, \hat{U}_i] \cdot \hat{L}^{-1}.$$

Here

$$\begin{pmatrix} 1 \dots m \\ n_1 \dots n_m \end{pmatrix}$$

is a permutation with $k_{n_i} = k_i$; $\hat{L}_1, \dots, \hat{L}_m$ and $\hat{L}$ are elements of $\hat{\mathfrak{G}}$.

A homeomorphism $N^* \to N^*$ is called **admissible** if it maps each branching vertex to a vertex of the same order.

Theorem 2. *Every automorphism of the group $\mathfrak{G}$ is induced by an admissible homeomorphism of the surface N^* onto itself.*

If $\mathfrak{G}$ is the fundamental group of a surface (i.e. $m = 0$), then every homeomorphism of this surface is admissible. Consequently, we obtain Nielsen' s theorem.

Other proofs of his theorem were given by H. Seifert (^{5}) and W. Mangler (^{6}) with the aid of Hopf' s theory (^{7}) on deformations of continuous mappings between surfaces. Our theorems were proved by W. Vollmerhaus (^{3}) in the case $g = 0$ and $k_i \geqslant 5$.

§ 3. Let

$$\hat{K}_1 = \hat{L}_1 \hat{S}_{i_1}^{\varepsilon_1} \hat{L}_1^{-1}, \dots, \hat{K}_m = \hat{L}_m \hat{S}_{i_m}^{\varepsilon_m} \hat{L}_m^{-1}, \hat{K}_{m+1}, \dots, \hat{K}_{m+2g}$$

be arbitrary elements of $\hat{\mathfrak{G}}$ ($\varepsilon_i = \pm 1$). By $\Pi_{\hat{K}}$ we denote a formal word in $\hat{K}_1, \dots, \hat{K}_{m+2g}$, in which each symbol $\hat{K}_i$ ($1 \leqslant i \leqslant m$) occurs exactly once and each symbol $\hat{K}_j$ ($j > m$) occurs twice, to the powers $+1$ and -1. Then $\{\hat{K}_1, \dots, \hat{K}_{m+2g}; \Pi_{\hat{K}}\}$ is called an **alternating product**.** $\Pi_{\hat{K}}(\hat{K}_1, \dots, \hat{K}_{m+2g})$, in contrast to $\Pi_{\hat{K}}$, denotes an element of $\hat{\mathfrak{G}}$.

If in $\Pi_{\hat{K}}$ there is a part $\hat{K}_i^{\varepsilon} \hat{K}_j^{\eta}$ with $i > m$, $i \neq j$, and $\varepsilon, \eta = \pm 1$, then take the elements $\hat{L}_i^{\varepsilon} = \hat{K}_i^{\varepsilon} \hat{K}_j^{\eta}$ and $\hat{L}_t = \hat{K}_t$ for $t \neq i$. We change $\Pi_{\hat{K}}$ in a suitable way and obtain a new alternating product. If $i \leqslant m$, then put $\hat{L}_i = \hat{K}_j^{-\eta} \hat{K}_i \hat{K}_j^{\eta}$ and $\hat{L}_t = \hat{K}_t$. These processes, and the corresponding substitutions of the first m and the last $2g$ elements for the other side, and transitions from $\hat{K}_i$ to $\hat{K}_i^{-1}$, are called **elementary**.

Two alternating products are regarded as **equivalent** if one can pass from one to the other by elementary processes. An alternating product has the **Nielsen property** if no element in it (as a word in the generators) cancels completely with both neighbors or with one more than halfway, and if more than half of the first element remains in the expression $\Pi_{\hat{K}}$. If only the second condition is satisfied, then the alternating product is called **reduced**.

* If $g = 0$, then we require that all $k_i \geqslant 5$ (see Vollmerhaus' s paper (^{3}), for which these conditions were needed). Already from the fact alone that $\hat{\alpha}$ is an automorphism carrying the kernel of the standard homomorphism into itself, the equalities of Theorem 1 (^{4}) are often obtained.

** This generalizes the concept from (^{2}), where the case $m = 0$ is considered. The proofs of paper (^{2}) carry over to the most general case.

To every alternating product there belongs a related product with the Nielsen property.

The assertion of equivalence of Theorems 1 and 2 is also carried over: if the alternating product $\{\hat{K}_1, \dots, \hat{K}_{m+2g}; \Pi_{\hat{K}}\}$ satisfies the equality

$$\Pi_{\hat{K}}(\hat{K}_1, \dots, \hat{K}_{m+2g}) = \hat{S}_1 \cdots \hat{S}_m \prod [\hat{T}_i, \hat{U}_i],$$

then it is related to $\{\hat{S}_1, \dots, \hat{T}_1, \hat{U}_1, \dots; \hat{S}_1 \cdots \hat{S}_m \prod [\hat{T}_i, \hat{U}_i]\}$. Hence it follows, in particular, that $\hat{K}_1, \dots, \hat{K}_{m+2g}$ are free generators of the group $\hat{\mathfrak{G}}$.

Alternating products are obtained from cuts of the surfaces N^*. Elementary processes are induced by replacements of cuts or by homeomorphisms. This is described in (^{2}) for $m = 0$, and for $g = 0$ it is a well-known fact from the theory of braids (see, for example, (^{8})). Thus, from Theorem 1 it follows that every automorphism of the group $\mathfrak{G}$ (up to an inner one) is induced by an admissible homeomorphism $N^* \to N^*$. It is easy to prove that inner automorphisms are also induced by homeomorphisms; hence Theorem 2 follows from Theorem 1. Conversely, Theorem 1 is obtained from Theorem 2—this is the basic fact from the theory of closed surfaces and spheres with holes.

§ 4. Let α be an automorphism of the group $\mathfrak{G}$. Since only the elements $LS_j^a L^{-1}$ have finite order, α can be induced by an endomorphism $\hat{\alpha}$ with

$$\hat{\alpha}\hat{S}_i = \hat{M}_i \hat{S}_{n_i}^{\alpha_i} \hat{M}_i^{-1}.$$

By Π_0 we denote the word

$$\hat{S}_1 \cdots \hat{S}_m \prod_{i=1}^{g} [\hat{T}_i, \hat{U}_i].$$

We regard $\hat{\alpha}\Pi_0$ in the group graph as a path with arbitrary initial vertex. After free reduction in $\hat{\alpha}\Pi_0$ we obtain another path, which is called **proper**. $\hat{\alpha}\Pi_0$ and its proper path are closed, but, generally speaking, not simply closed. If Π' is a proper part of Π_0, then $\hat{\alpha}\Pi'$ is not closed. The most important place in the proof of Theorem 1 is occupied by the following

Lemma 1. *Suppose that all $a_i = 1$. If the proper path for $\hat{\alpha}\Pi_0$ is not simply closed, then α can be induced by such an endomorphism $\hat{\beta}$ of the group $\hat{\mathfrak{G}}$ that the length of the proper path will be smaller.*

The idea of the proof is as follows: we consider the alternating product

$$\{\hat{\alpha}\hat{S}_1, \dots, \hat{\alpha}\hat{U}_g; \hat{\alpha}\Pi_0\}$$

and pass to the related reduced product

$$\{\hat{K}_1, \dots, \hat{K}_{m+2g}; \Pi_{\hat{K}}\},$$

where, for $i \leq m$,

$$\hat{K}_i = \hat{L}_i \hat{S}_{p_i} \hat{L}_i^{-1}.$$

To every symbol from $\hat{L}_i$ $(i \leq m)$ and $\hat{K}_j$ $(j > m)$ there belongs its formally inverse symbol from $\hat{L}_i^{-1}$ or $\hat{K}_j^{-1}$, which is called the **inverse partner**. In addition, the symbols that do not remain in the proper path for $\Pi_{\hat{K}}$ have **cancelling partners**. We construct chains from inverse and cancelling partners. They are either closed or have ends on the proper path. Let $\hat{W}$ be the reduced word for

$$\Pi_{\hat{K}}(\hat{K}_1, \dots, \hat{K}_{m+2g})$$

and let $\hat{W} = \hat{W}_1 \hat{W}_2$ be a decomposition of the proper path into two loops. If there is a chain with ends $\hat{X}_1, \hat{X}_2$ on $\hat{W}_1 = \hat{V}\hat{X}_1\hat{V}'$ and $\hat{W}_2$, then we replace, in the words $\hat{K}_1, \dots, \hat{K}_{m+2g}$, the symbols of this chain by the elements $\hat{V}^{-1}\hat{V}'^{-1}$ or $\hat{V}'\hat{V}$. Since $\hat{W}_1$ is a relation of the group $\mathfrak{G}$, the new elements $\hat{K}'_1, \dots, \hat{K}'_{m+2g}$ induce in $\mathfrak{G}$ the same elements that were induced by the elements $\hat{K}_1, \dots, \hat{K}_{m+2g}$. But the length of the proper path decreases by two or more.

A chain with ends on both loops exists if a double point separates on the proper path at least one pair $\hat{K}_i, \hat{K}_i^{-1}$. The alternating product

$$\{\hat{\alpha}\hat{S}_1, \dots, \hat{\alpha}\hat{U}_g; \hat{\alpha}\Pi_0\}$$

has a related reduced ⋯

product with this property. Having reversed all the necessary elementary processes, we obtain an endomorphism $\hat{\beta}$, which induces α.

Thus, one can induce α by such an endomorphism $\hat{\gamma}$ that the proper path for $\hat{\gamma}\Pi_0$ is a simple closed contour. If we knew that it is the boundary of one disk in the planar group picture, then we would obtain the last equation of Theorem 1 and hence everything that is required.

§ 5. We shall consider the kernel $\mathfrak{N}^*$ of the standard homomorphism.

Lemma 2. Let $\mathfrak{Z}^{m+1}$ be a free commutative group with generators $e_0, e_1, \dots, e_m$. The mapping

$$\hat{L}\hat{S}_i^{k_i}\hat{L}^{-1} \to e_i \quad \text{and} \quad \hat{L}\Pi_0\hat{L}^{-1} \to e_0$$

for arbitrary $\hat{L} \in \hat{\mathfrak{G}}$ determines a homomorphism $\tau : \mathfrak{N} \to \mathfrak{Z}^{m+1}$.

This is proved by means of the Reidemeister–Schreier method (^{9}) and the representation of the group $\mathfrak{G}$ as a free product with amalgamated subgroups*.

Every endomorphism $\hat{\alpha} : \hat{\mathfrak{G}} \to \hat{\mathfrak{G}}$ which induces an endomorphism of the group $\mathfrak{G}$ determines an endomorphism of the group $\mathfrak{Z}^{m+1}$ by the formulas

$$e_0 \to \tau\hat{\alpha}\Pi_0, \qquad e_j \to \tau\hat{\alpha}\hat{S}_j^{k_j}.$$

In its matrix $M(\hat{\alpha})$, every row except the first contains only one number different from zero.

Lemma 3. If the endomorphism $\hat{\varepsilon}$ induces the identity automorphism of the group $\mathfrak{G}$, then $M(\hat{\varepsilon})$ is the identity matrix.

In the proof one must distinguish the cases $g = 0$ (^{3}), $m = 0$, and $m, g > 0$. It follows from the lemma that for every endomorphism over an automorphism of the group $\mathfrak{G}$ the first row of the matrix has the form $(1, 0, \dots, 0)$. If the proper path for $\gamma\Pi_0$ is a simple closed contour, then it is the boundary of a subcomplex, and $\tau\hat{\gamma}\Pi_0$ shows how many disks it contains with boundaries of the types $\Pi_0, \hat{S}_1^{k_1}, \dots, \hat{S}_m^{k_m}$.

Thus,

$$\hat{\gamma}\Pi_0 = \hat{M}\Pi_0\hat{M}^{-1},$$

as was required to prove.

Moscow State University
named after M. V. Lomonosov

Received
30 X 1963

REFERENCES

[1] J. Nielsen, Acta Math., **50**, 189 (1927).
[2] H. Zieschang, Abhandl. math. Sem. Univ. Hamburg, **27**, No. 1/2 (1964).
[3] W. Vollmerhaus, Über die Automorphismen ebener Gruppen, Dissertation, Göttingen, 1963.
[4] M. Greendlinger, Arch. Math., **12**, 94 (1961).
[5] H. Seifert, Abhandl. math. Sem. Univ. Hamburg, **12**, 29 (1938).
[6] W. Mangler, Math. Zs., **44**, 541 (1939).
[7] H. Hopf, J. reine u. angew. Math., **163**, 71 (1930); **165**, 225 (1931).
[8] G. Burde, Math. Ann., **151**, 101 (1963).
[9] M. Hall, Theory of Groups, IL, 1962.

[*] Let, for example, $\mathfrak{A}$ be the group with generators $S_1, \dots, S_m$ and defining relations $S_1^{k_1} = \cdots = S_m^{k_m} = 1$, and let $\mathfrak{B}$ be the free group with generators $T_1, \dots, U_g$. Then $\mathfrak{G}$ is the free product of the groups $\mathfrak{A}$ and $\mathfrak{B}$ with amalgamated subgroups, which are generated by the elements $(S_1 \cdots S_m)^{-1}$ and $\Pi[T_i, U_i]$.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.