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Abstract

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MATHEMATICS

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ON AN APPROXIMATE SOLUTION OF THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION

(Presented by Academician P. S. Novikov on 23 XII 1963)

In the unit s -dimensional cube G_s ($0 \leq x_i \leq 1$, $i = 1, 2, \dots, s$) with boundary Γ , the Dirichlet problem for the Laplace equation is considered

$$\Delta u = 0, \quad u|_{\Gamma} = \varphi(\sigma), \quad (1)$$

where the function $\varphi(\sigma) = \varphi(x_1, \dots, x_s)|_{\Gamma}$ is continuous on Γ .

The aim of the present work is an approximate solution of problem (1). It is known that at an arbitrary interior point P the solution of problem (1) can be represented in the form

$$u(P) = - \int_{\Gamma} \varphi(\sigma) \frac{\partial}{\partial v_{\sigma}} K(P, \sigma) d\sigma, \quad (2)$$

where $\frac{\partial}{\partial v_{\sigma}} K(P, \sigma)$ is the derivative of the Green's function along the normal to Γ . To obtain the solution of the Dirichlet problem for the Laplace equation, we shall approximately compute the integral (2). The main difficulty here will consist in obtaining a convenient formula for the Green's function.

Introduce the following notation: $f(x_1, \dots, x_s) \in H_s^{\alpha}$ ($\alpha > 1$ an integer), if the derivative $\partial^{\alpha s} f(x_1, \dots, x_s) / \partial x_1^{\alpha} \dots \partial x_s^{\alpha}$ is continuous in G_s ; $f(x_1, \dots, x_s) \in E_s^{\alpha}(c)$ ($\alpha > 1$), if

$$f(x_1, \dots, x_s) = \sum_{m_1, \dots, m_s = -\infty}^{\infty} C(m_1, \dots, m_s) e^{2\pi i(m_1 x_1 + \dots + m_s x_s)},$$

$$|C(m_1, \dots, m_s)| \leq \frac{C}{\bar{m}_1^\alpha \dots \bar{m}_s^\alpha},$$

where $\bar{m}_\nu = \max(1, |m_\nu|)$.

Define the function $\tau(x)$ by the equality

$$\tau(x) = (2\alpha - 1)C_{2(\alpha-1)}^{\alpha-1} \left(\frac{x^\alpha}{\alpha} - \frac{C_{\alpha-1}^1}{\alpha+1}x^{\alpha+1} + \dots \pm \frac{C_{\alpha-1}^{\alpha-1}}{2\alpha-1}x^{2\alpha-1} \right)$$

and denote by $\varphi_\nu(x_1, \dots, x_{\nu-1}, x_{\nu+1}, \dots, x_s)$ the difference

$$\varphi(x_1, \dots, x_{\nu-1}, 0, x_{\nu+1}, \dots, x_s) - \varphi(x_1, \dots, x_{\nu-1}, 1, x_{\nu+1}, \dots, x_s)$$

and by R the error of the approximate solution of problem (1).

Let $Q(\xi_1, \dots, \xi_s)$ be any point of G_s ; $P(x_1, \dots, x_s)$ an arbitrary interior point. Let

$$\min_{1 \leq i \leq s} \min(x_i, 1 - x_i) \geq \delta > 0.$$

Introduce the function $F^{[n]}(P, Q)$

$$F^{[n]}(P, Q) = \begin{cases} \frac{1}{\omega_s(s-2)} \left[r_{PQ}^{2-s} - \rho^{2-s} - \beta_1 (r_{PQ}^2 - \rho^2) - \dots - \beta_n (r_{PQ}^{2n} - \rho^{2n}) \right], & r_{PQ} \leq \rho, \\ 0, & r_{PQ} \geq \rho. \end{cases}$$

where $\omega_s = \frac{2\pi^{s/2}}{\Gamma(s/2)}$; $\rho \leq \delta$; the constants $\beta_1, \beta_2, \dots, \beta_n$ are chosen from the condition of continuity, for $r_{PQ} = \rho$, of derivatives up to order n of the function $F^{[n]}(PQ)$.

Lemma. Let $J_\nu(x)$ be the Bessel function of order ν ; $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$ the eigenvalues of the Laplace operator for the first boundary-value problem in G_s ; $u_1(x), u_2(x), \dots, u_n(x), \dots$ the corresponding eigenfunctions. Then, for $s \geq 3$ and $n > s/2 - \frac{3}{2}$, for the Green function $K(P, Q)$ the equality

$$K(P, Q) = F^{[n]}(P, Q) + C(n, s) \sum_{i=1}^{\infty} \frac{u_i(P)u_i(Q)J_{s/2+n-1}(\rho\sqrt{\lambda_i})}{\lambda_i^{s/4+n/2+1/2}}, \quad (3)$$

holds, where

$$C(n, s) = \frac{2^{s/2+n-1} \Gamma(n + 1/2)}{\rho^{s/2+n-1}}.$$

The proof of the lemma follows from expanding $F^{[n]}(P, Q)$ in a Fourier series in the eigenfunctions of the Laplace operator (see (1)) and comparing the series obtained with the Fourier series of the Green function. We note that, putting

$$F^{[n]}(P, Q) = \begin{cases} \frac{1}{2\pi} \left[\ln \frac{1}{r_{PQ}} - \ln \frac{1}{\rho} - \beta_1 (r_{PQ}^2 - \rho^2) - \dots - \beta_n (r_{PQ}^{2n} - \rho^{2n}) \right], & r_{PQ} \leq \rho, \\ 0, & r_{PQ} \geq \rho, \end{cases}$$

we shall have the lemma for the case $s = 2$.

Let a_1, a_2, \dots, a_{s-1} be $(s-1)$ -dimensional optimal coefficients modulo N (see (2), p. 96). Define the parameters n_1 and μ by the relations

$$\frac{1}{a^2 s} (n_1 + as - s/2 + 1)^2 \ln(s + 2n_1 - 2) = \ln N,$$

$$N^\alpha \mu^{-n_1/2+s/4-1/2} = n_1. \quad (4)$$

Theorem. If $\varphi(\sigma) \in H_{s-1}^\alpha$ on each $(s-1)$ -dimensional unit cube constituting Γ , then, for any $\varepsilon > 0$, for the solution of problem (1) at an arbitrary interior point P the equality

$$\begin{aligned} u(P) &= \frac{B(n, s)}{N} \sum_{k=1}^N \sum_{\nu=1}^s \sum_{\substack{m_i \geq 1 \\ 1 \leq (m_1^2 + \dots + m_s^2) \leq \mu}} \frac{m_\nu J_{s/2+n-1} \left[\rho \pi (m_1^2 + \dots + m_s^2)^{1/2} \right] \prod_{i=1}^s \sin \pi m_i x_i}{(m_1^2 + \dots + m_s^2)^{s/4+n/2+1/2}} \\ &\times \varphi_\nu \left[\tau \left(\left\{ \frac{ka_1}{N} \right\} \right), \dots, \tau \left(\left\{ \frac{ka_{\nu-1}}{N} \right\} \right), \tau \left(\left\{ \frac{ka_\nu}{N} \right\} \right), \dots, \tau \left(\left\{ \frac{ka_{s-1}}{N} \right\} \right) \right] \times \prod_{i=1}^{s-1} \tau' \left(\left\{ \frac{a_i k}{N} \right\} \right) \\ &\times \prod_{l=\nu+1}^{s'} \prod_{j=1}^{\nu-1} \sin \pi m_j \tau \left(\left\{ \frac{a_j k}{N} \right\} \right) \sin \pi m_l \tau \left(\left\{ \frac{a_{l-1} k}{N} \right\} \right) + O \left(\frac{1}{N^{\alpha-\varepsilon}} \right), \end{aligned} \quad (5)$$

where

$$B(n, s) = -\frac{2^s C(n, s)}{\pi^{s/2+n}}, \quad n = [n_1].$$

Proof. Starting from formulas (2) and (3), for $n > s/2 - \frac{1}{2}$ we have:

$$u(P) = -C(n, s) \int_{\Gamma} \sum_{i=1}^{\infty} \frac{u_i(P) \varphi(\sigma) \frac{\partial}{\partial \nu_{\sigma}} u_i(\sigma)}{\lambda_i^{s/4+n/2+1/2}} J_{s/2+n-1}(\rho \sqrt{\lambda_i}) d\sigma. \quad (6)$$

The eigenvalues and eigenfunctions of the Laplace operator for the first boundary-value problem in G_s are:

$$\lambda_i = \lambda_{m_1, \dots, m_s} = \pi^2(m_1^2 + \dots + m_s^2);$$

$$u_i(x_1, \dots, x_s) = u_{m_1, \dots, m_s}(x_1, \dots, x_s) = 2^{s/2} \prod_{i=1}^s \sin \pi m_i x_i.$$

Substituting these formulas into (6) and restricting ourselves to a finite number of terms of the series, and taking into account that $|J_{s/2+n-1}(\rho \sqrt{\lambda_i})| \leq 1$, we obtain:

$$\begin{aligned} u(P) = B(n, s) & \int_0^1 \dots \int_0^1 \sum_{\nu=1}^s \sum_{\substack{m_i \geq 1 \\ 1 \leq (m_1^2 + \dots + m_s^2) \leq \mu}} \frac{m_{\nu} J_{s/2+n-1}[\rho \pi (m_1^2 + \dots + m_s^2)^{1/2}]}{(m_1^2 + \dots + m_s^2)^{n/2+s/4+1/2}} \\ & \times \prod_{i=1}^s \sin \pi m_i x_i \cdot \varphi_{\nu}(\xi_1, \dots, \xi_{\nu-1}, \xi_{\nu+1}, \dots, \xi_s) \prod_{\substack{i=1 \\ i \neq \nu}}^s \sin \pi m_i \xi_i d\xi_i \\ & + O\left(\frac{B(ns)}{\sqrt{\nu} \mu^{n/2+1/4}}\right). \end{aligned} \quad (7)$$

In formula (7) we make the change of variables $\xi_i = \tau(\xi'_i)$ ($i = 1, 2, \dots, s$, $i \neq \nu$). It is easy to show that the integrand in the right-hand side of (7) will belong to the class $E_{s-1}^{\alpha}[B(n, s)]$. The error of computing the integral of this function by the method of optimal coefficients is equal to:

$$O\left(\frac{B(n, s) \ln^{\gamma} N}{N^{\alpha}}\right),$$

where γ depends on α and s .

Thus, for the error of the approximate solution of problem (1) the estimate

$$R = O\left(\frac{B(n, s)}{\sqrt{\nu} \mu^{n/2+1/4}}\right) + O\left(\frac{B(n, s) \ln^{\gamma} N}{N^{\alpha}}\right)$$

is valid.

From the estimate

$$\sum_{\substack{m_i \geq 1 \\ 1 \leq m_1^2 + \dots + m_s^2 \leq \mu}} 1 = O(\mu^{s/2})$$

and from the definition of the parameters n_1 and μ in (4), for $n = [n_1]$ it follows that

$$\sum_{\substack{m_i \geq 1 \\ 1 \leq m_1^2 + \dots + m_s^2 \leq \mu}} 1 = O(N^{\gamma_1/\sqrt{\ln N}}) = O(N^\varepsilon),$$

$$R = O\left(\frac{(\ln N)^{\gamma_2\sqrt{\ln N}}}{\rho^{\sqrt{\ln N}}N^\alpha}\right) + O\left(\frac{(\ln N)^{\gamma_3\sqrt{\ln N}}}{\rho^{\sqrt{\ln N}}N^\alpha}\right) = O\left(\frac{1}{N^{\alpha-\varepsilon_1}}\right),$$

where $\gamma_1, \gamma_2, \gamma_3$ depend only on α and s ; $\varepsilon, \varepsilon_1$ are arbitrarily small positive numbers. The theorem is proved.

Remark 1. Differentiating equality (2), we obtain a derivative of any order of the solution of problem (1). Since the point P is an interior point, the right-hand side of (2) may be differentiated under the integral sign. It follows from the proof of the theorem that on the classes H_{s-1}^α a derivative of any order of the solution of the Dirichlet problem for the Laplace equation at an arbitrary interior point can be computed by the method of optimal coefficients with error

$$O\left(\frac{1}{N^{\alpha-\varepsilon}}\right),$$

where ε is an arbitrarily small positive number.

Remark 2. An analogous theorem holds for computing the solution of problem (1) by the method of uniform grids on the classes D_{s-1}^α (see (2), p. 31). The question of the minimal order of the number of elementary operations required for solving problem (1) was considered by N. S. Bakhvalov (3). From the results of (3) it follows that, on the classes D_{s-1}^α and H_{s-1}^α , the error estimate for computing the solution of the Dirichlet problem for the Laplace equation in formula (5) does not admit any substantial improvement.

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Note: Figure translations are in progress. See original paper for figures.

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