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Abstract

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ON THE BEHAVIOR OF SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR A QUASILINEAR PARABOLIC EQUATION AS $t \to \infty$

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1. In the present note we study the behavior, as $t \to \infty$, of the solution of the parabolic equation

$$Lu \equiv \frac{D}{Dx} f(x, u, u_x) - a(x, t, u, u_x) \frac{\partial u}{\partial t} = 0 \tag{1}$$

(the symbol D/Dx denotes total differentiation with respect to x) for the boundary-value problem:

$$u(x, 0) = u_0(x), \qquad u(0, t) = 0, \qquad u_x(l, t) = A_0, \tag{2}$$

and also for the first boundary-value problem:

$$u(x, 0) = u_0(x), \qquad u(0, t) = \varphi_1(t), \qquad u(l, t) = \varphi_2(t) \tag{3}$$

under the condition that $\varphi_i(t) \to \varphi_i^0$ as $t \to \infty$ $(i = 1, 2)$.

Suppose that, for all u and p and $(x, t) \in D\{0 \le x \le l,\ 0 \le t < \infty\}$, the following conditions are satisfied:

$$\frac{\partial f(x, u, p)}{\partial p} \ge \alpha > 0, \tag{4}$$

$$0 < a(x, t, u, p) \le \gamma_1(x, u, p), \tag{5}$$

where α is a constant and $\gamma_1(x, u, p)$ is a certain continuous function.

It is natural to expect that the solution $u(x, t)$, as $t \to \infty$, tends to the solution $z(x)$ of the ordinary differential equation

$$\frac{D}{Dx} f(x, z, z_x) = 0, \tag{6}$$

or, what is the same thing, to the solution $z(x, c)$ of the equation

$$f(x, z, z_x) = c \tag{6'}$$

for some fixed value of the parameter c.

2. In considering the boundary-value problem (1), (2), we shall assume that the following conditions are fulfilled:

a) the function $f(x, u, p)$ is six times continuously differentiable with respect to x, u, p; the function $a(x, t, u, p)$ is five times continuously differentiable with respect to x, t, u, p (for $(x, t) \in D$ and arbitrary u and p);

b) there exists a solution $u(x, t)$ of problem (1), (2), and the derivative $u_x(x, t)$ is bounded in the domain $D_T\{0 < x < l,\ 0 < t < T\}$ for every $T > 0$.

Suppose now that, for the given A_0, there exists a solution $v(x)$ of equation (6) satisfying the conditions: $v(0) = 0$, $v'(l) = A_0$. In this case, on the interval $[0, l]$, for all c from some neighborhood (c_1, c_2) of the point $c_0 \equiv f(x, v(x), v'(x))$, there exists a solution $z(x, c)$ of the Cauchy problem for the equa-

···of equation $(6')$ with the condition $z(0, c) = 0$. Suppose, moreover, that

$$z_{xc}(l, c_0) > 0 \tag{7}$$

and that the initial function $u_0(x)$ satisfies the inequalities

$$z(x, \tilde{c}_1) \le u_0(x) \le z(x, \tilde{c}_2) \tag{8}$$

(here $\tilde{c}_1, \tilde{c}_2 \in (c_1, c_2)$ and are such that $z_{xc}(l, c) > 0$ for $\tilde{c}_1 < c < \tilde{c}_2$).

Theorem 1. *If conditions (4), (5), a), b), (7), and (8) are satisfied, then the solution of problem (1), (2) satisfies $u(x, t) \to v(x)$ as $t \to \infty$, uniformly in x.*

Proof. Denote by $w_i(x, t)$ the solutions of equation (1) satisfying the conditions: $w_i(x, 0) = z(x, \tilde{c}_i)$, $w_i(0, t) = 0$,

$$\frac{\partial}{\partial x} w_i(l, t) = A_i(t) \quad (i = 1, 2).$$

The functions $A_i(t) \in C^6[0, \infty)$ and are subject to the conditions:

$$A_1'(t) \ge 0, \qquad A_2'(t) \le 0, \qquad \lim_{t\to\infty} A_i(t) = A_0, \qquad A_i(t) \equiv z_x(l, \tilde{c}_i)$$

for $0 \le t \le \delta$ $(\delta > 0)$.

We shall show that the following inequalities hold in the domain D:

$$w_1(x,t) \le u(x,t) \le w_2(x,t), \qquad z(x,\tilde{c}_1) \le w_i(x,t) \le z(x,\tilde{c}_2), \tag{9}$$

$$\frac{\partial}{\partial t} w_1(x,t) \ge 0, \qquad \frac{\partial}{\partial t} w_2(x,t) \le 0 \quad (i = 1, 2).$$

Represent equation (1) in the form

$$a(x,t,u,u_x)\frac{\partial u}{\partial t} = b(x,u,u_x)\frac{\partial^2 u}{\partial x^2} + d(x,u,u_x)\frac{\partial u}{\partial x} + c(x,u)u + g(x), \tag{1'}$$

where

$$b(x,u,u_x) = f_p(x,u,u_x), \quad d(x,u,u_x) = \int_0^1 f_{xp}(x,u,\tau u_x)\,d\tau + f_u(x,u,u_x),$$

$$c(x,u) = \int_0^1 f_{xu}(x,\tau u,0)\,d\tau, \quad g(x) = f_x(x,0,0).$$

Since the functions $w_i(x,t)$, $u(x,t)$, and $z(x,\tilde{c}_i)$ $(i = 1,2)$ are solutions of equation $(1')$, the difference of any two of them, as well as the functions $\frac{\partial}{\partial t} w_i(x,t)$, may be regarded as solutions of certain linear homogeneous parabolic equations with coefficients continuous inside D.

In order to obtain inequalities (9), it is enough to use Theorems 6 and 7 of [1].

We next show that in the domain D the estimate

$$\left|\frac{\partial w_i}{\partial x}\right| \le M, \tag{10}$$

holds, where M is some constant.

Integrating the identity $Lw_i = 0$ with respect to x from x to l, we obtain the relation:

$$\int_x^l a\left(x,t,w_i,\frac{\partial}{\partial x}w_i\right)\frac{\partial w_i}{\partial t}\,dx = f(l,w_i(l,t),A_i(t)) - f\left(x,w_i,\frac{\partial}{\partial x}w_i\right). \tag{11}$$

In view of the boundedness of

$$\left|f\left(0, 0, \frac{\partial}{\partial x} w_i(0,t)\right)\right|$$

and the inequality

$$\left|\int_x^l a\,\frac{\partial w_i}{\partial t}\,dx\right| \leq \left|\int_0^l a\,\frac{\partial w_i}{\partial t}\,dx\right|,$$

from (11) it follows that there is a bound, uniform in x and t in the domain D, ...

of the functions $f\left(x, w_i, \frac{\partial}{\partial x} w_i\right)$ $(i = 1, 2)$. Estimate (10) now follows from inequalities (4) and (9). Denote

$$\lim_{t\to\infty} w_i(x,t) = \mu_i(x).$$

Since the integral

$$\int_0^t \int_0^l a\left(x, t, w_i, \frac{\partial}{\partial x} w_i\right) \frac{\partial w_i}{\partial t}\,dx\,dt$$

is bounded uniformly in t for $t \geqslant 0$, for each i $(i = 1, 2)$ there exists a sequence of numbers $\{t_n^i\}$ such that

$$\int_0^l a\,\frac{\partial w_i}{\partial t}\,dx \to 0 \quad \text{as } t_n^i \to \infty.$$

Hence it follows that

$$\int_x^l a\,\frac{\partial w_i}{\partial t}\,dx \to 0 \quad \text{as } t_n^i \to \infty$$

uniformly with respect to x. Taking this into account, from relation (11) we obtain:

$$f\left(x, w_i(x, t_n^i), \frac{\partial}{\partial x} w_i(x, t_n^i)\right) \to f\left(x, \mu_i(l), A_0\right) \tag{12}$$

as $t_n^i \to \infty$, uniformly with respect to x.

The Arzelà theorem may be applied to the sequence of functions $\{w_i(x, t_n^i)\}$; by virtue of it, for each i $(i = 1, 2)$ there exists a subsequence of numbers $\{t_{n_k}^i\}$ such that

$$w_i(x, t_{n_k}^i) \to \mu_i(x)$$

as $t_{n_k}^i \to \infty$, uniformly with respect to x for $0 \leqslant x \leqslant l$. It then follows from conditions (4) and (12) that the sequence of functions

$$\left\{\frac{\partial}{\partial x} w_i(x, t_{n_k}^i)\right\}$$

converges uniformly in x as $t^i_{n_k} \to \infty$.

Thus $\mu_i(x)$ are solutions of equation (6), satisfying the conditions:

$$\mu_i(0) = 0, \qquad \mu_i'(l) = A_0.$$

Since $z(x, c)$ is a monotonically increasing function of c for a fixed value $x \in (0, l]$, it follows that

$$\mu_i(x) \equiv v(x) \quad (i = 1, 2).$$

The theorem is proved.

Remark 1. In the proof of Theorem 1, the existence of solutions $w_i(x, t)$ and their smoothness were assumed (continuity of

$$\frac{\partial}{\partial t} w_i(x, t)$$

in the domain D, and the existence of

$$\frac{\partial^2}{\partial t^2} w_i(x, t), \qquad \frac{\partial^3}{\partial t\, \partial x^2} w_i(x, t)$$

inside D, $i = 1, 2$). Applying the method of proof of Theorem 1 from paper ([2]), one can show that the solutions $w_i(x, t)$ exist and possess the smoothness indicated above.

Remark 2. In the particular case when equation (1) takes the form

$$\frac{\partial^2 u}{\partial x^2} = A(x, t, u)\frac{\partial u}{\partial t} + B(x, u)\frac{\partial u}{\partial x} + F(x, u),$$

assumption b) will be valid if, for all values of u and $(x, t) \in D$, the following conditions are satisfied: the functions $A(x, t, u)$, $B(x, u)$, $F(x, u)$ have continuous derivatives of third order with respect to all arguments;

$$0 < A_1(x, t) \leqslant A(x, t, u) \leqslant A_2(x, t),$$

$$F_u(x, u) \geqslant M_1, \qquad |B_u(x, u)| + |F_{uu}(x, u)| \leqslant M_2;$$
$$u_0(x) \in C^3[0, l], \qquad u_0(0) = 0, \qquad u_0'(l) = A_0$$

(here $A_i(x, t)$ are certain continuous functions, M_i are constants, $i = 1, 2$).

Theorem 1, in essence, asserts that condition (7) is sufficient for the stability of the stationary solution of equation (1). It is proved analogously that from the condition $z_{xc}(l, c_0) < 0$ there follows instability of the stationary solution. If, however, $z_{xc}(l, c_0) = 0$, then the stationary solution may be either stable or unstable.

We now consider the case when, for a given A_0, there exist two solutions of equation (6) satisfying the conditions:

$$z(0) = 0, \qquad z'(l) = A_0.$$

Denote these solutions by $z_1(x)$ and $z_2(x)$. Suppose that for all $c \in [c_1, c_2]$ (where $c_i = f(x, z_i, z_i')$, $i = 1, 2$) on the interval $[0, l]$ there exists a solution $z(x, c)$ of the Cauchy problem for equation (6′) with the condition $z(0, c) = 0$, and moreover let

$$z_x(l, c) < A_0 \tag{13}$$

for $c_1 < c < c_2$. Let $u(x, t)$ denote the solution of problem (1), (2) under the condition that $u_0(x)$ satisfies the inequalities:

$$z(x, c_1) < u_0(x) \leq z(x, c_2), \qquad z_x(0, c_1) < u_0'(0) \quad (0 < x \leq l). \tag{14}$$

Theorem 2. *If conditions (4), (5), a), b), (13), and (14) are fulfilled, then*

$$u(x, t) \to z_2(x) \quad \text{as } t \to \infty$$

uniformly in x.

3. In studying the behavior of the solution of equation (1) in the case of the first boundary-value problem, we shall assume that the following conditions are fulfilled:

c) the existence of solutions $v_1(x)$ and $v_2(x)$ of equation (6) such that

$$\min_{0 \leq x \leq l} v_1(x) \geq \max\left\{\max_{0 \leq x \leq l} u_0(x), \sup_{0 \leq t < \infty} \varphi_i(t)\right\},$$

$$\max_{0 \leq x \leq l} v_2(x) \leq \min\left\{\min_{0 \leq x \leq l} u_0(x), \inf_{0 \leq t < \infty} \varphi_i(t)\right\} \qquad (i = 1, 2);$$

uniqueness of the solution $v_0(x)$ satisfying the conditions:

$$v_0(0) = \varphi_1^0, \qquad v_0(l) = \varphi_2^0;$$

d) the existence of a solution $u(x, t)$ of equation (1) with derivative $u_{xx}(x, t)$ bounded in the domain D_T (for every $T > 0$) for any sufficiently smooth initial and boundary conditions (3), and its smoothness: continuity of $u_t(x, t)$ in D and existence of u_{tt}, u_{xxt} inside D in the case when $u_0(x)$ is a solution of equation (6).

In addition, we shall assume that instead of condition (5) the following condition is fulfilled:

$$0 < a(x, t, u, p) \leq \gamma_2(x, u) \tag{5'}$$

(where $\gamma_2(x, u)$ is some continuous function).

Theorem 3. *If conditions (4), (5′), c), and d) are fulfilled, then the solution of problem (1), (3)*

$$u(x,t) \to v_0(x) \quad \text{as } t \to \infty$$

uniformly in x.

Remark 3. Sufficient for the fulfillment of condition c) is the following condition:

$$|f(x,u,0)| \leq M_1 + M_2|u|, \qquad M_i > 0, \quad 0 \leq x \leq l, \quad |u| < \infty \ (i = 1, 2).$$

Applying Theorem 1 of work ([3]), one can indicate additional (very cumbersome) conditions on the functions $a(x,t,u,p)$, $f(x,u,p)$, and $u_0(x)$ under which assertion d) will be valid. The behavior of solutions of boundary-value problems for quasilinear parabolic equations of another type in the case of an arbitrary number of variables was investigated in work ([4]).

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REFERENCES

1. A. M. Il' in, A. S. Kalashnikov, O. A. Oleinik, UMN, 17, issue 3 (105), 10–13 (1962).

2. Chjou Yu-lin, Sci. Sinica, 10, No. 8, (1961).

3. A. F. Filippov, DAN, 141, No. 4 (1961).

4. S. N. Khudyaev, Some investigations on nonlinear parabolic and elliptic equations, Dissertation, Moscow State University, 1963.

Note: Figure translations are in progress. See original paper for figures.