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Abstract

Full Text

MATHEMATICS

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SOME EXTREMAL PROBLEMS IN THE CLASS OF MAPPINGS QUASICONFORMAL ON THE AVERAGE

(Presented by Academician M. A. Lavrentiev on 30 XI 1963)

It is known that the class of quasiconformal mappings with characteristic $p \leq q = \text{const} < \infty$, i.e., the class of q -quasiconformal mappings, is compact, and various extremal problems are solved naturally in it. It is of interest to consider extremal problems in the class of mappings quasiconformal on the average, for which the integral $\iint \Phi(p) d\sigma_z$ is bounded, where Φ is some function. In what follows, unless otherwise specified, quasiconformality is understood in this sense.

We shall consider mappings $w = f(z)$ of the disk $|z| \leq 1$ onto the disk $|w| \leq 1$. Let $\Phi(x)$, $1 \leq x \leq \infty$, be a continuously differentiable function and let $\Phi'(x) > 0$, and let $F(w_1, w_2, \dots, w_n)$ be a real-valued function of the variables $w_k = f(z_k) = u_k + iv_k$, continuously differentiable with respect to u_k, v_k .

Problem 1. In the class of quasiconformal mappings of the disk $|z| \leq 1$ onto the disk $|w| \leq 1$, $w(0) = 0$, $w(1) = 1$, with bounded integral

$$J = \iint_{|z| \leq 1} \Phi(p) d\sigma_z \leq M = \text{const} < \infty$$

find a mapping for which the function F assumes its maximum value.

Suppose that an extremal mapping giving the maximum of the function F exists. Subjecting the disk $|w| \leq 1$ to a variation with constant characteristic h in the disk $K(\xi, \tau) : |w - \xi| \leq \tau$ (see ⁽¹⁾), we find that the increment of the integral J is equal to

$$\delta J = \pi \tau^2 J_{z/w}(\xi) \Phi'(p_\xi) \delta p, \quad (1)$$

where $\delta p = -2p|h| \cos 2(\theta - \varphi)$, $\varphi = \frac{1}{2} \arg h + \frac{1}{2} \pi$, and $J_{z/w}, p, \theta$ are, respectively, the Jacobian and the characteristics of the mapping inverse to $w = f(z)$. The increment of the function F can be represented in the form

$$dF = -\frac{2\tau^2}{\pi} |A| \cdot |h| \cos(\varphi - \theta_A), \quad (2)$$

where

$$A(\xi) = \sum_{k=1}^n \left[\frac{F_{w_k} w_k (1 - w_k)}{\xi(1 - \xi)(w_k - \xi)} + \frac{\bar{F}_{w_k} \bar{w}_k (1 - \bar{w}_k)}{\xi(1 - \xi)(1 - \bar{w}_k \xi)} \right], \quad \theta_A = -\frac{1}{2} \arg A(\xi)$$

(we do not consider the case $A \equiv 0$, corresponding to the presence of a stationary value of F ; $F_{w_k} = 0$, $k = 1, 2, \dots, n$).

Comparing (1) and (2), we see that if $J < M$, then a variation with any sufficiently small h is admissible and dF can have any sign. Therefore, in the case of an extremal mapping the following conditions must be satisfied:

$$\iint_{|z| \leq 1} \Phi(p) d\sigma_z = M, \quad \theta(w) = -\frac{1}{2} \arg A(w).$$

Moreover, the extremal mapping must possess the following property:

$$\frac{|A|}{p^{J_{z/N}} \Phi'(p)} = c = \text{const.}$$

We note that in the general case the extremal mapping may have discontinuities at the marked points z_k . This is most easily seen in the case when one point is marked.

Problem 2. In the class of quasiconformal mappings of the annulus $r \leq |z| \leq 1$ onto annuli $\rho^* = |w| \leq 1$, $r < \rho \leq \rho^* < 1$, find a mapping for which the integral

$$\iint_{r \leq |z| \leq 1} p(z) d\sigma_z$$

takes its minimum value.

This problem is also solved by means of the variational method. By a variation of the annulus $r \leq |z| \leq 1$ we shall mean a function $w = f(z)$ mapping it onto some annulus $r' \leq |w| \leq 1$, normalized and such that the mapping effected by it is conformal outside some disk $K(\zeta, \tau) : |z - \zeta| \leq \tau$, while inside it has a prescribed constant characteristic h .

In this case the equality

$$\delta r = r' - r = r\tau^2 \operatorname{Re} \frac{h}{\zeta^2}$$

holds.

Varying the annulus in the w -plane, we find that the extremal mapping will be the mapping onto the annulus $\rho \leq |w| \leq 1$, and the characteristics of the inverse mapping satisfy the conditions:

$$\theta(w) = \arg w + \pi/2, \quad pJ_{z/w}|w^2| = c = \text{const.}$$

The following assertions are valid:

I. If $\rho \leq e^{r-1}$, then the extremal mapping is

$$w = \begin{cases} \frac{1}{c} e^{c|z|-1+i(\alpha+\arg z)}, & r \leq |z| \leq \frac{1}{c}, \\ ze^{i\alpha}, & \frac{1}{c} < |z| \leq 1, \end{cases}$$

where c is the smaller root of the equation $\rho ec = e^{cr}$, and α is a real constant.

II. If $\rho > e^{r-1}$, then the extremal mapping is

$$w = \rho^{\frac{|z|-1}{r-1}} e^{i\psi(\arg z)},$$

where $\psi = \psi(x)$ is any continuously differentiable function carrying the interval $[0, 2\pi]$ into an interval of length 2π and such that $\psi'(x) \geq 1/\ln \frac{1}{\rho}$.

The solution obtained makes it possible to formulate the following theorem.

Theorem 1. Let the annulus $r \leq |z| \leq 1$ be mapped quasiconformally onto the annulus $\rho \leq |w| \leq 1$, and let

$$\iint_{r \leq |z| \leq 1} p(z) d\sigma_z \leq M.$$

Then:

1) if $M \leq 2\pi(1-r)$, then

$$\rho \leq \frac{1}{c} e^{cr-1}, \quad \text{where } \frac{1}{c} = r + \sqrt{\frac{M}{\pi} + r^2 - 1};$$

2) if $M > 2\pi(1-r)$, then

$$\rho \leq e^{\frac{2\pi}{M}(r-1)}.$$

Equality is attained only for mappings of the form

$$w = \begin{cases} \frac{1}{c} e^{c|z|-1+i(\alpha+\arg z)}, & r \leq |z| \leq \frac{1}{c}, \\ ze^{i\alpha}, & \frac{1}{c} < |z| \leq 1. \end{cases}$$

(in the first case) and $w = e^{\frac{2\pi}{M}(|z|-1)+i\psi(\arg z)}$ (in the second case), where a is a real constant, $\psi = \psi(x)$ is any continuously differentiable function mapping the interval $[0, 2\pi]$ onto an interval of length 2π and such that $\psi'(x) \geq 2\pi/M$.

Problem 3. In the class of quasiconformal mappings of the annulus $r \leq |z| \leq 1$ onto annuli $\rho^* \leq |w| \leq 1$, $0 < \rho^* \leq \rho < r$, find a mapping for which the integral

$$\iint_{r \leq |z| \leq 1} p(z) d\sigma_z$$

assumes its minimum value.

Solving this problem in the subclass of q -quasiconformal mappings, we obtain that the extremal mapping is of the form

$$w = \begin{cases} \rho \left(\frac{|z|}{r} \right)^q e^{i(\alpha+\arg z)}, & r \leq |z| \leq r^{\frac{q}{q-1}} \rho^{\frac{1}{1-q}}, \\ ze^{i\alpha}, & r^{\frac{q}{q-1}} \rho^{\frac{1}{1-q}} < |z| \leq 1. \end{cases}$$

Although the limit of the value of the integral as $q \rightarrow \infty$ exists, there is no mapping for which this limiting value would be attained.

Theorem 2. Suppose the annulus $r \leq |z| \leq 1$ is mapped quasiconformally onto the annulus $\rho \leq |w| \leq 1$, with

$$\iint_{r \leq |z| \leq 1} p(z) d\sigma_z \leq M.$$

Then

$$\rho > r e^{\frac{\pi(1-r^2)-M}{\pi r^2}}.$$

Let us note that the class of mappings quasiconformal in the mean is not equicontinuous. For example, if in Theorem 1 we pass to the limit as $r \rightarrow 0$, and then conformally map the annulus $\rho \leq |w| \leq 1$, obtained under the extremal mapping, onto the disk $|w| \leq 1$ with a radial slit from 0 to R , then we obtain a mapping of the disk onto the disk which sends the point $z = 0$ into a segment of length $R = R(M)$, where

$$R(M) = \begin{cases} \lambda \left(\frac{1}{e} \sqrt{\frac{M}{\pi}} - 1 \right), & \text{if } M \leq 2\pi, \\ \lambda \left(e^{-\frac{2\pi}{M}} \right), & \text{if } M > 2\pi; \end{cases}$$

here $\lambda(x)$ is the inverse function of

$$x = e^{-\frac{\pi}{2} \frac{K'(\lambda)}{K(\lambda)}},$$

and $K(\lambda)$ and $K'(\lambda)$ denote the complete elliptic integrals of the first kind corresponding respectively to the moduli λ and $\lambda' = \sqrt{1 - \lambda^2}$.

It is clear that the magnitude of the gap $R(M)$ is maximal for the given class of mappings. In particular, for quasiconformal mappings of the disk $|z| \leq 1$ onto the disk $|w| \leq 1$, $w(0) = 0$, for which

$$\iint_{|z| \leq 1} (p(z) - 1) d\sigma_z \leq \varepsilon,$$

we obtain, accurate up to small terms of order $\varepsilon^{3/2}$,

$$R(\varepsilon) = \frac{4}{e} \sqrt{\frac{\varepsilon}{\pi}}.$$

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CITED LITERATURE

1. P. P. Belinskii, DAN, **121**, No. 2 (1958).

Note: Figure translations are in progress. See original paper for figures.

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