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# G. M. ZASLAVSKII, B. V. CHIRIKOV

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Fig. 1

Figure 1: Fig. 1

**Abstract****Full Text****G. M. ZASLAVSKII, B. V. CHIRIKOV****ON THE MECHANISM OF FERMI ACCELERATION IN THE ONE-DIMENSIONAL CASE***(Presented by Academician M. A. Leontovich, 11 V 1964)*

For explaining the origin of cosmic rays, Fermi <sup>(1)</sup> proposed a statistical mechanism of particle acceleration upon their collisions with macroscopic cosmic clouds, by analogy with molecular collisions. Recently <sup>(2, 3)</sup>, the fundamental possibility has been considered of using an analogous mechanism for accelerating space rockets in the gravitational field of planets or stars. However, the question of the criteria of stochasticity remains open, i.e., the conditions under which statistical laws are applicable to a mechanical system moving along a completely definite trajectory (especially if the system consists of a small number of bodies, for example a rocket in the field of a double star <sup>(3)</sup>).

**Fig. 1**

In work <sup>(4)</sup> the simplest case of Fermi acceleration was investigated—the motion of a light particle between two parallel planes, one of which oscillates according to a prescribed law. A numerical calculation of the motion of such a particle led to a negative result: acceleration was practically not observed. The particle velocity sometimes reached 3–4 wall velocities and in most cases was of the order of the wall velocity, whereas, according to the Fermi mechanism, the mean particle velocity should grow without bound proportionally to time <sup>(5)</sup>.

In the present work a criterion of stochasticity is obtained, and a detailed further investigation is carried out of the mechanism of Fermi acceleration in the one-dimensional case.

Let us consider the following dynamical system: a particle moves between two plane infinitely heavy walls, with which it collides according to the laws of perfectly elastic impact (Fig. 1). One wall is fixed, the other oscillates strictly periodically, and during each half-period its velocity changes linearly with time. The motion of the particle is described by the following exact system of difference equations:

$$v_{n+1} = \pm v_n + V(\psi_n - 1/2); \quad (1)$$

$$\psi_{n+1} = 1/2 - 2v_{n+1}/V + \sqrt{(1/2 - 2v_{n+1}/V)^2 + 4\varphi_n v_{n+1}/V} \quad (v_{n+1} > V\psi_n/4); \quad (2)$$

$$\psi_{n+1} = 1 - \psi_n + 4v_{n+1}/V \quad (v_{n+1} \leq V\psi_n/4); \quad (3)$$

$$\varphi_n = \{\psi_n + [\psi_n(1 - \psi_n) + l/4a]/(4v_{n+1}/V)\}; \quad (4)$$

$v_n$  is the particle velocity;  $n$  is the number of the collision with the moving wall;  $V/4$  is the amplitude of the wall velocity;  $\psi_n$  is the phase of the oscillating wall at the instant of impact, varying from 0 to 1/2 when the wall moves from position  $A$  to  $B$  and from 1/2 to 1 during the reverse motion; the braces  $\{\dots\}$  denote the fractional part of the argument. The plus sign in (1) corresponds to formula (2) at the preceding step, and the minus sign to formula (3).

As will be seen from what follows, the interesting case is

$$l/a \gg 1, \quad v_n/V \gg 1. \quad (5)$$

Then the system of equations takes the form:

$$\Delta v(n) \equiv v_{n+1} - v_n = V(\psi_n - 1/2), \quad \psi_{n+1} \simeq \varphi_n \simeq \{\psi_n + lV/16av_{n+1}\}. \quad (6)$$

If, in addition,

$$lV/16av_{n+1} \ll 1, \quad (7)$$

then system (6) can be approximately replaced by the differential equations

$$v'(n) = V\left(\psi(n) - \frac{1}{2}\right), \quad \psi'(n) = lV/16av(n), \quad (8)$$

where the prime denotes differentiation with respect to  $n$ , which plays the role of a dimensionless time. System (8) is conveniently investigated with the aid of the so-called phase equation:

$$\Phi'' + \Omega^2\Phi = 0, \quad (9)$$

where  $\Phi = \psi - \frac{1}{2}$  and  $\Omega^2 = lV^2/16av^2(n) \ll 1$ .

It follows from (5) that  $v$  changes only slightly, and to first approximation  $\Omega$  may be regarded as constant. Then from (8), (9) it follows that the particle velocity executes small oscillations of amplitude  $\sim v_0\sqrt{a/l}$  about the initial value  $v_0$ , with frequency  $\Omega$ . We note that fulfillment of inequality (7) corresponds to an adiabatic oscillation of the wall relative to the oscillation of the particle.

In work <sup>(6)</sup> a system analogous to (8) was investigated, and it was shown that under the condition

$$\Omega^2 \gg 1 \quad (10)$$

the change in  $v$  becomes stochastic. Application of the stochasticity criterion (10) to the original system (6) is hindered by criterion (7), which is violated when criterion (10) is fulfilled. Therefore we shall obtain the stochasticity criterion by another method. To this end, consider the correlation of the phases  $\psi_n$  and  $\psi_{n+1}$ . We define the correlation coefficient by the formula:

$$\rho = \left\langle \left( \psi_{n+1} - \frac{1}{2} \right) \left( \psi_n - \frac{1}{2} \right) \right\rangle / \left\langle \left( \psi_n - \frac{1}{2} \right)^2 \right\rangle, \quad (11)$$

where  $\langle \dots \rangle$  denotes averaging over the phase ( $\psi_n$ ). Taking (6) into account, we obtain\*

$$\rho = \int_0^1 \left( \psi - \frac{1}{2} \right) \left[ \{\theta - k\psi\} - \frac{1}{2} \right] d\psi / \int_0^1 \left( \psi - \frac{1}{2} \right)^2 d\psi, \quad (12)$$

where  $k = lV^2/16av^2 - 1 = \Omega^2 - 1$ ,  $\theta = (lV/16av)(1 + V/2v)$ .

For  $\Omega^2 \ll 1$

$$\rho \approx 1 - 6\theta(1 - \theta) \sim 1, \quad (13)$$

i.e., there is a strong correlation between neighboring phases and stochasticity is absent. For  $\Omega^2 \gg 1$

$$\rho \approx \frac{1}{k} [1 - 6\theta(1 - \theta)] \rightarrow 0. \quad (14)$$

In this case the correlations are practically absent, and this means that  $v$  changes stochastically.

The preceding consideration shows that there exist three qualitatively different regions of particle velocities: I.  $v \lesssim \frac{1}{4}V\sqrt{l/a}$ . II.  $\frac{1}{4}V\sqrt{l/a} < v \ll Vl/16a$ . III.  $v \gg Vl/16a$ . In region I the Fermi mechanism operates; in region III the particle velocity executes stable small oscillations; region II is intermediate.

Fig. 2.  $l/a \simeq 10^4$

Figure 2: Fig. 2.  $l/a \simeq 10^4$

During the motion the particle may pass from region I to II and back; region III is completely isolated.

Let us consider region I in more detail. Since the motion of the particle in I is stochastic, one can introduce the velocity distribution function  $f(v)$  and write a kinetic equation for it. For small changes in the particle velocity as a result of collisions ( $V/v \ll 1$ ), the kinetic equation was obtained in (7). In our case, taking into account that one of the “particles” —the wall—is infinitely heavy, the kinetic equation takes the form

$$\frac{\partial f(v, t)}{\partial t} = \frac{1}{2} \frac{\partial}{\partial v} \left( D(v) \frac{\partial f(v, t)}{\partial v} \right), \quad (15)$$

\* In computing the correlation coefficient by formula (11), we assume ergodicity of the motion, i.e., a uniform distribution of the phases  $\psi_n$  on the interval  $(0, 1)$  and its independence of the velocity. In our case, ergodicity apparently takes place under the condition  $\theta \gg 1$ .

where  $D(v)$  is the “diffusion” coefficient, defined by the formula

$$D(v) = \langle (\Delta v)^2 / \Delta t \rangle. \quad (16)$$

Here  $\Delta v$  is the change in velocity upon collision (6), and  $\Delta t$  is the time between two collisions. From equations (2), (4),

$$\Delta t = 2l/v. \quad (17)$$

Fig. 2.  $l/a \simeq 10^4$

Case (3) is excluded by inequalities (5). Substitution of (17), (6) into (16) gives

$$D(v) = vV^2/24l. \quad (18)$$

In solving equation (15), as a boundary condition we take

$$D(v) \partial f / \partial v \Big|_{v=v_1} = 0. \quad (19)$$

Condition (19) means the absence of flux at  $v = v_1$ , i.e., the cessation of further acceleration of the particle. It is natural to take as the boundary

$$v_1 \sim \frac{1}{4}V\sqrt{l/a}, \quad (20)$$

since the probability of penetration of the particle into region II is small because of the violation of stochasticity there. We take the initial conditions in the form

$$f(v, 0) = \delta(v - v_0). \quad (21)$$

The solution of equation (15) under conditions (19), (21) is

$$f(v, t) = \frac{1}{v_1} \sum_n e^{-\lambda_n \tau} J_0(2\sqrt{\lambda_n v_0}) J_0(2\sqrt{\lambda_n v}) / J_0^2(2\sqrt{\lambda_n v_1}), \quad (22)$$

where  $J$  is the Bessel function;  $\tau = Vt/48l$ ;  $\lambda_n$  are the roots of the equation  $J_1(2\sqrt{\lambda_n v_1}) = 0$ . As  $t \rightarrow \infty$ ,

$$f(v, t) \simeq \frac{1}{v_1} \left( 1 + e^{-\lambda_1 t} \frac{J_0(2\sqrt{\lambda_1 v_0}) J_0(2\sqrt{\lambda_1 v})}{J_0^2(2\sqrt{\lambda_1 v_1})} \right), \quad \lambda_1 \simeq \frac{3.68}{v_1}, \quad (23)$$

and the distribution function tends to the constant value  $1/v_1$  with relaxation time

$$t_r = 13lv_1/V^2. \quad (24)$$

For  $t \ll t_r$ , condition (19) is unimportant, and the solution with  $v_0 = 0$ ,

$$f(v, t) = e^{-v/\tau}/\tau \quad (25)$$

coincides with that obtained in <sup>(5)</sup> under the additional condition  $v \gg V$ . In particular, the mean velocity is

$$\bar{v} = \tau, \quad t \ll t_r; \quad \bar{v} = v_1/2, \quad t \gg t_r. \quad (26)$$

To check the assumptions made, the system (1)–(4) was solved on an electronic computer for the following parameter values:  $a = 1$ ,  $V \simeq 4$ ,  $v_0 = 0$ . Several variants with different  $l$ ,  $\psi_0$  were computed for  $N = 10^5$  collisions in each. To reduce errors associated with the finite number of digits of the machine, the mantissas of the quantities  $l$  and  $V$  were chosen as a set of random numbers. The result of the calculation was the distribution function  $F(v, t)$ , defined as the fraction of the total time of motion  $t$  spent by the particle in a given velocity interval. The relation between  $f$  and  $F$  is given by the formula

Fig. 3

Figure 3: Fig. 3

$$F(v, t) = \frac{1}{t} \int_0^t f(v, t) dt. \quad (27)$$

Figure 2 shows a typical distribution function  $F(v)$  for  $t \gg t_r$ . The flat (up to fluctuations) part of the distribution function corresponds to region I. Along the abscissa is plotted the particle velocity in units of the wall's maximum velocity. The arrow marks the maximum velocity attained by the particle after  $10^5$  collisions. The fluctuations of the distribution function in region I are determined by the number  $n$  of independent passages of the particle through the entire acceleration region. The latter is approximately equal to the ratio of the total acceleration time  $t \simeq 2lN/v_1$  to the relaxation time (24). For the fluctuations we obtain the estimate

$$|\Delta f/f| \sim 1/\sqrt{n} \sim \sqrt{l/3aN}. \quad (28)$$

For  $l/a = 10^4$  (Fig. 2),  $|\Delta f/f| \sim 1/6$ .

Fig. 3.  $1-l/a \approx 400$ ;  $2-l/a \approx 10^4$ ;  $3-l/a \approx 4 \cdot 10^4$

Figure 3 illustrates the fulfillment of the stochasticity criterion (10). As is seen from the graphs, the boundary between regions I and II lies at  $\Omega^{-1} = 4(v_1/V)\sqrt{a/l} = 1/2$ , which corresponds to estimate (20). We note that the particle penetrates rather far (especially for small  $l$ ) into region II, but does not reach region III, whose boundary  $\Omega_{\text{III}}^{-1} \sim \frac{1}{4}(l/a)^{3/2} \sim 2000$  ( $l = 400$ ) lies at very large velocities.

The absence of acceleration obtained in (4) is apparently explained by the fact that the authors took a small ratio  $l/a$ , which leads only to an insignificant excess of the particle velocity over the wall velocity (20).

In conclusion we note that the case of two (and a larger number of) dimensions differs fundamentally from the one-dimensional case, as was demonstrated by the example in paper (8). The difference is connected with the fact that, in the case of several degrees of freedom, stochasticity of the particle motion can be established as a result of redistribution of energy among the degrees of freedom. Then  $V \sim \Delta v \sim v$  (large-angle scattering), and consequently the stochasticity criterion (10) is apparently always satisfied, and no stationary distribution exists.

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