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Abstract

Full Text

MATHEMATICS

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DESCRIPTION OF EXTENSIONS OF HERMITIAN POSITIVE FUNCTIONS

(Presented by Academician S. N. Bernstein on 12 VI 1964)

As is known, a Hermitian positive (h.p.) function $F(x; y)$, given in the strip $-\infty < x < \infty$, $|y| \leq 2a$, can always be extended to the whole plane with preservation of Hermitian positivity*. However, the question of describing all h.p. extensions remains open. In ⁽³⁾, by methods of operator theory, the problem of describing h.p. extensions was solved in the case when $F(x; 0)$ is an almost periodic function. Below a description of all extensions in the general case is given. In character our methods are close to the methods of ⁽⁴⁾. At the same time, an independent proof is obtained of the possibility of extending an h.p. function given in a strip.

1. Let $F(x; y)$ be a continuous h.p. function given in the strip $-\infty < x < \infty$, $|y| \leq 2a$. Form the function

$$\sigma(\lambda; y) - \sigma(0; y) = \text{l.i.m.} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\lambda x} - 1}{-ix} F(x; y) dx. \quad (1)$$

Lemma. The function $\sigma(\lambda; y) - \sigma(0; y)$ is continuous in y and Hermitian monotone in λ , i.e. $\sigma(\lambda''; y) - \sigma(\lambda'; y)$ is h.p. for $\lambda'' > \lambda'$, $|y| \leq 2a$. Moreover,

$$|\sigma(\lambda''; y) - \sigma(\lambda'; y)| \leq \sigma(\lambda''; 0) - \sigma(\lambda'; 0),$$

where $\sigma(\lambda) = \sigma(\lambda; 0)$.

Consider the linear set K_{2a} of functions of the form

$$x(t) = c + \int_{-2a}^{2a} e^{ity} \varphi(y) dy, \quad \varphi(y) \in L^1(-2a, 2a).$$

On K_{2a} define a linear functional by putting

$$F_\lambda[x(t)] = c\sigma(\lambda; 0) + \int_{-2a}^{2a} \sigma(\lambda; y) \varphi(y) dy.$$

We shall show that $F_{\lambda''} - F_{\lambda'}$ is nonnegative on K_{2a} for $\lambda'' > \lambda'$. Indeed, let $x(t) \geq 0$. Then the function $x_1(t) = \gamma + x(t)$, $\gamma > 0$, on the real axis satisfies the inequality $\inf x_1(t) \geq \gamma > 0$. As shown in the article ⁽⁵⁾, the function $x_1(t)$ is representable in the following form:

$$x_1(t) = \left| \sqrt{\gamma} + \int_{-a}^a e^{ity} \psi(y) dy \right|^2, \quad \psi(y) \in L^1(-a, a).$$

* This result was obtained by M. S. Livshits as a consequence of one general theorem of operator theory. It was included in the doctoral dissertation of M. S. Livshits, defended in 1946, but was not published. Independently this result was obtained by A. Devinatz ⁽¹⁾ and G. I. Eskin ⁽²⁾.

** The existence of the integral follows from the inequality $|F(x; y)| \leq F(0; 0)$. The function $\sigma(\lambda; y)$ is normalized by the conditions

$$\sigma(\lambda; y) = \frac{\sigma(\lambda + 0; y) + \sigma(\lambda - 0; y)}{2}, \quad \sigma(-\infty; y) = 0.$$

From this representation and the lemma it follows that $(F_{\lambda'} - F_{\lambda})[x_1(t)] \geq 0$. Passing to the limit as $\gamma \downarrow 0$, we obtain that $(F_{\lambda'} - F_{\lambda})[x(t)] \geq 0$, i.e. $\{F_{\lambda}\}$ is a monotone family of functionals. From the monotonicity of $\{F_{\lambda}\}$ and the equality $F_{\lambda}[1] = \sigma(\lambda; 0)$, $-\infty < \lambda < \infty$, it follows that $\{F_{\lambda}\}$ are uniformly bounded. Extend $\{F_{\lambda}\}$ by continuity to the space B_{2a} —the closure of the set K_{2a} in the uniform metric*. Since every nonnegative function in B_{2a} is the uniform limit of nonnegative functions from K_{2a} , the family $\{F_{\lambda}\}$ remains monotone after extension. Let h_n be some sequence of positive numbers monotonically tending to zero. Consider the difference quotient

$$\psi_{\lambda h_n}(x) = \frac{F_{\lambda+h_n}(x) - F_{\lambda}(x)}{\sigma(\lambda + h_n) - \sigma(\lambda)}, \quad \text{if } \sigma(\lambda + h_n) - \sigma(\lambda) \neq 0;$$

$$\psi_{\lambda h_n}(x) = 0, \quad \sigma(\lambda + h_n) - \sigma(\lambda) = 0.$$

The norm of the functional $\psi_{\lambda h_n}(x)$ is attained on the function $x(t) \equiv 1$, and since $\psi_{\lambda h_n}(1)$ is either one or zero, $\|\psi_{\lambda h_n}\| \leq 1$. The unit sphere in B_{2a}^* is weakly compact, and therefore from the sequence $\psi_{\lambda h_n}(x)$ one can extract a weakly convergent subsequence $\psi_{\lambda h'_n}(x)$. Put

$$\psi_{\lambda} = \lim_{h'_n \rightarrow 0} \psi_{\lambda h'_n}.$$

The functional ψ_{λ} is positive and $\|\psi_{\lambda}\| \leq 1$. By a known theorem of M. G. Krein ⁽⁶⁾, every functional ψ_{λ} can be extended, preserving positivity, to the space C^{∞} . The extended functionals $\tilde{\psi}_{\lambda}(x)$ admit the representation

$$\tilde{\psi}_\lambda(x) = \int_{-\infty}^{\infty} x(t) d\rho_\lambda(t) + \mu_\lambda x'(\infty) **.$$

With the help of the nondecreasing functions $\rho_\lambda(t)$ we construct the functions $\psi(\lambda; y)$, setting

$$\psi(\lambda; y) = \int_{-\infty}^{\infty} e^{ity} d\rho_\lambda(t).$$

Obviously,

$$\psi(\lambda; y) = \lim_{h \rightarrow 0} \int_{-\infty}^{\infty} \left(\frac{1}{h} \int_0^{y+h} e^{its} ds \right) d\rho(\lambda; t),$$

and since the integrand function belongs to K_{2a} , $\psi(\lambda; y)$ does not depend on the choice of extension of the functionals. From what follows it will follow that the functions $\psi(\lambda; y)$ are uniquely determined by the p.d. function $F(x; y)$.

We shall call the functions $\psi(\lambda; y)$ the **canonical functions** of the p.d. function $F(x; y)$. Since $\rho_\lambda(t)$ is a nondecreasing function of bounded variation, $\psi(\lambda; y)$ are continuous p.d. functions of y .

We show how the canonical functions are found and how $F(x; y)$ is expressed through them. Denote, for $k > 0$,

$$\sigma^k(\lambda; y) = \frac{1}{k} \int_0^k \sigma(\lambda; y + t) dt.$$

The inequality

$$|\sigma^k(\lambda + h; y) - \sigma^k(\lambda; y)| \leq \sigma(\lambda + h; 0) - \sigma(\lambda; 0), \quad h > 0,$$

shows that the function $\sigma^k(\lambda; y)$ is absolutely continuous with respect to the measure $\sigma(\lambda) = \sigma(\lambda; 0)$. Therefore, almost everywhere with respect to the measure $\sigma(\lambda)$

* Elements of the space B_{2a} are entire functions of degree $\leq 2a$ having a limit at ∞ .

** It can be shown that σ -almost everywhere $\mu_\lambda = 0$.

there exists $\varphi_k(\lambda; y) = d\sigma^k(\lambda; y)/d\sigma(\lambda)$ and

$$\sigma^k(\lambda; y) = \int_0^\lambda \varphi_k(\lambda; y) d\sigma(\lambda) + \sigma^k(0; y).$$

Denoting

$$\psi^k(\lambda; y) = \frac{1}{k} \int_0^k \psi(\lambda; y + t) dt,$$

we see that

$$\psi^k(\lambda; y) = \int_{-\infty}^{\infty} e^{ity} \frac{e^{ikt} - 1}{kt} d\rho_\lambda(t) = \psi_\lambda \left[e^{ity} \frac{e^{ikt} - 1}{kt} \right].$$

Since $e^{ity}(e^{ikt} - 1)/(kt) \in K_{2a}$, putting $x(t) = e^{ity}(e^{ikt} - 1)/(kt)$, we find that for fixed y

$$\begin{aligned} \psi^k(\lambda; y) &= \lim_{h_n \rightarrow 0} \frac{F_{\lambda+h_n}(x) - F_\lambda(x)}{\sigma(\lambda + h_n) - \sigma(\lambda)} = \lim_{h_n \rightarrow 0} \frac{\sigma^k(\lambda + h_n; y) - \sigma^k(\lambda; y)}{\sigma(\lambda + h_n) - \sigma(\lambda)} = \frac{d\sigma^k(\lambda; y)}{d\sigma(\lambda)} \\ &= \varphi_k(\lambda; \varphi) \end{aligned}$$

for σ -almost all values of λ . Hence

$$\sigma^k(\lambda; y) - \sigma^k(0; y) = \int_0^\lambda \psi^k(\lambda; y) d\sigma(\lambda).$$

Since the functions $\sigma^k(\lambda; y) - \sigma^k(0; y)$ and $\psi^k(\lambda; y)$ are continuous in y ($|y| \leq 2a$), as $k \rightarrow 0$ we shall have

$$\sigma^k(\lambda; y) - \sigma^k(0; y) \rightarrow \sigma(\lambda; y) - \sigma(0; y)$$

and

$$\psi^k(\lambda; y) \rightarrow \psi(\lambda; y).$$

Moreover, $|\psi^k(\lambda; y)| \leq 1$. Passing to the limit under the integral sign, we obtain

$$\sigma(\lambda; y) - \sigma(0; y) = \int_0^\lambda \psi(t; y) d\sigma(t). \quad (2)$$

Equality (2) shows that for fixed y , σ -almost everywhere

$$\psi(\lambda; y) = d\sigma(\lambda; y)/d\sigma(\lambda),$$

i.e. the channel functions are essentially uniquely determined by the positive-definite function $F(x; y)$. Since $\psi(\lambda; y)$ is a continuous function of y , the latter equality can be used for computing the channel functions. Substituting (2) into (1) and then reversing equality (1), we obtain:

Theorem 1. Let $F(x; y)$ be a continuous positive-definite function, defined in the strip

$$-\infty < x < \infty, \quad |y| \leq 2a;$$

let $\sigma(\lambda)$ be the measure specifying the representation of the function

$$F(x; 0) = \int_{-\infty}^{\infty} e^{i\lambda x} d\sigma(\lambda).$$

Let $\psi(\lambda; y)$ be the channel functions of the positive-definite function $F(x; y)$. Then the function $F(x; y)$ has the representation

$$F(x; y) = \int_{-\infty}^{\infty} e^{i\lambda x} \psi(\lambda; y) d\sigma(\lambda), \quad -\infty < x < \infty, \quad |y| \leq 2a. \quad (3)$$

Conversely, if $\sigma(\lambda)$ is a bounded monotone function on $(-\infty, \infty)$, and $\psi(\lambda; y)$ is positive-definite, continuous in y ($|y| \leq 2a$) for every fixed λ , measurable for every y with respect to $\sigma(\lambda)$, and σ -almost everywhere $\psi(\lambda; 0) = 1$, then $F(x; y)$ in representation (3) is a continuous positive-definite function in the strip $|y| \leq 2a$, and $\psi(\lambda; y)$ are its channel functions.

Thus, extension of the positive-definite function $F(x; y)$ from the strip $|y| \leq 2a$ to the whole plane with preservation of positive-definiteness is equivalent to simultaneous extension of the channel functions $\psi(\lambda; y)$ so that the extensions of the positive-definite function $\psi(\lambda; y)$ form, for fixed $|y|$, a σ -measurable function of λ . Such extensions we shall call **admissible**.

2. We shall describe all admissible extensions. Without loss of generality, one may assume that, except possibly for a set of σ -measure zero, all functions $\psi(\lambda; y)$ extend nonuniquely.* In this case the extensions are described in a well-known way (7). With the aid of the function

* In the opposite case one may take the product $e^{-\varepsilon|y|}\psi(\lambda; y)$ and pass to the limit as $\varepsilon \rightarrow 0$.

For $\psi(\lambda; y)$ one introduces the form

$$(f, \bar{g})_\lambda = \int_{-a}^a \int_{-a}^a \psi(\lambda; t-s) f(t) \bar{g}(s) dt ds.$$

For fixed functions f and g the form $(f, \bar{g})_\lambda$ is σ -measurable, since the functions $\psi(\lambda; y) = \partial\sigma(\lambda; y)/\partial\sigma(\lambda)$ are σ -measurable. From the form $(f, \bar{g})_\lambda$ four special entire functions $P_1(z; \lambda)$, $P_2(z; \lambda)$, $Q_1(z; \lambda)$, $Q_2(z; \lambda)$ of degree $\leq 2a$ are constructed. From the method of constructing these functions it follows that they are σ -measurable in λ . Form the function

$$R(z; \lambda) = \frac{P_1(z; \lambda) + \tau(z; \lambda)Q_1(z; \lambda)}{P_2(z; \lambda) + \tau(z; \lambda)Q_2(z; \lambda)} = \int_{-\infty}^{\infty} \frac{d\rho(t; \lambda)}{t-z},$$

where the function $\tau(z; \lambda)$ is an R -function in z^* and is σ -measurable in λ . From the Stieltjes inversion formula it follows that $\rho(t; \lambda)$ is a nondecreasing function of bounded variation in t and is σ -measurable in λ . From the latter it follows immediately that the functions

$$\tilde{\psi}(\lambda; y) = \int_{-\infty}^{\infty} e^{ity} d\rho(t; \lambda)$$

are e.p. extensions of the canonical functions; moreover, $\tilde{\psi}(\lambda; y_0)$ is σ -measurable in λ for every y_0 .

By the method described above all e.p. extensions $F(x; y)$ can be obtained. Indeed, an admissible extension $\tilde{\psi}(\lambda; y)$ of the canonical functions has the form

$$\tilde{\psi}(\lambda; y) = \int_{-\infty}^{\infty} e^{ity} d\rho(t; \lambda).$$

From the inversion formula one obtains the σ -measurability in λ of the function $\rho(t; \lambda)$. From the equality

$$\int_{-\infty}^{\infty} \frac{d\rho(t; \lambda)}{t-z} = \frac{P_1(z; \lambda) + \tau(z; \lambda)Q_1(z; \lambda)}{P_2(z; \lambda) + \tau(z; \lambda)Q_2(z; \lambda)},$$

where $\tau(z; \lambda)$ is an R -function in z for σ -almost all λ , the σ -measurability in λ of the function $\tau(z; \lambda)$ follows easily.

3. Let $\tilde{\psi}'(\lambda; y)$, $\tilde{\psi}''(\lambda; y)$ be admissible extensions of the canonical functions $\psi(\lambda; y)$ of an e.p. function $F(x; y)$. We shall call them **essentially distinct** if, for at least one y_0 , the functions $\tilde{\psi}'(\lambda; y_0)$ and $\tilde{\psi}''(\lambda; y_0)$ differ on a set of positive σ -measure.

Theorem 2. *In order that an e.p. function $F(x; y)$, $-\infty < x < \infty$, $|y| \leq 2a$, have more than one e.p. extension, it is necessary and sufficient that its canonical functions admit essentially distinct admissible extensions.*

In conclusion we consider it our pleasant duty to express gratitude to M. G. Krein for his attention and interest in the work.

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REFERENCES CITED

1. A. Devinatz, *Acta Math.*, **102**, 1-2, 109 (1959).
2. G. I. Eskin, *DAN*, **133**, No. 3, 540 (1960).
3. I. E. Ovcharenko, *Dokl. AN ArmSSR*, **38**, no. 5, 257 (1964).
4. M. G. Krein, *Uch. zap. Kuibyshevsk. ped. inst.*, **7**, 123 (1943).
5. B. Ya. Levin, *DAN*, **52**, No. 4, 291 (1946); *Distribution of zeros of entire functions*, 1956, p. 566.
6. N. I. Akhiezer, M. G. Krein. *On certain questions in the theory of moments*, Kharkov, 1938.
7. M. G. Krein, *DAN*, **26**, No. 1, 17 (1940).

* That is, a function holomorphic inside the half-plane $\text{Im } z \geq 0$ and mapping it into itself.

Note: Figure translations are in progress. See original paper for figures.

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