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Abstract

Full Text

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On Linear Superpositions of Continuously Differentiable Functions

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In the present note, relying on results of A. G. Vitushkin ([1]), two theorems are proved.

Theorem 1. *For arbitrary functions $p_m(x_1, x_2)$ continuous in the entire plane and functions $q_m(x_1, x_2)$ continuously differentiable in the entire plane ($m = 1, 2, \dots, N$), and for any domain D in the plane of the variables x_1, x_2, the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x_1, x_2) f_m(q_m(x_1, x_2)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is nowhere dense in the space of all functions continuous in the domain D with uniform convergence.

It is interesting to compare this result with A. N. Kolmogorov' s theorem ([2]) on the possibility of representing every continuous function of two variables by a superposition of the form

$$\sum_{i=1}^{5} f_i(\alpha_i(x) + \beta_i(y)),$$

where all functions are continuous, while $\{\alpha_i(x) + \beta_i(y)\}$ are fixed in advance.*

Theorem 2. *For arbitrary functions $p_m(x_1, x_2)$ continuous in the entire plane and functions $q_m(x_1, x_2)$ continuously differentiable in the entire plane ($m = 1, 2, \dots, N$), and for any domain D, there exist natural numbers ν and μ such that the polynomial $Q(x_1, x_2) = (x_1 + \nu x_2)^\mu$ is not equal in the domain D to any superposition of the form*

$$\sum_{m=1}^{N} p_m(x_1, x_2) f_m(q_m(x_1, x_2)),$$

where $\{f_m(t)\}$ are arbitrary bounded measurable functions.

Theorem 2 generalizes A. G. Vitushkin' s theorem ([1]) on the existence of an analytic function $F(x_1, x_2)$ not equal in the domain D to any superposition of the indicated form.

The following lemma is essentially proved, but not explicitly formulated, in ([1]).

Lemma 1. *For arbitrary continuous functions $p_m(x_1, x_2)$ and continuously differentiable functions $q_m(x_1, x_2)$ $(m = 1, 2, \dots, N)$, and for any domain D, one can:*

1) *fix a closed subset $G \subset D$, which is the union of a finite number of simply connected closed domains;*
2) *specify indices $1 \leq m_1 < m_2 < \cdots < m_n \leq N$;*
3) *select on the intervals $\{I_k = [\min_G q_{m_k}(x_1, x_2);\ \max_G q_{m_k}(x_1, x_2)]\}$ a finite set of points $t_{k,j} \in I_k$ $(k = 1, 2, \dots, n;\ j = 1, 2, \dots, r_k)$ such that two conditions are satisfied:*

a) *for arbitrary bounded measurable functions $\{\varphi_m(t)\}$ there exist bounded measurable functions $\{f_k(t)\}$ such that*

$$\sum_{k=1}^{n} p_{m_k}(x_1, x_2) f_k(q_{m_k}(x_1, x_2)) = \sum_{m=1}^{N} p_m(x_1, x_2) \varphi_m(q_m(x_1, x_2)) \quad \text{in } G;$$

*

It can even be shown that A. N. Kolmogorov' s construction permits one to construct the functions $\{\alpha_i(x)\}$ and $\{\beta_i(y)\}$ in his theorem so that they satisfy a Hölder condition with any exponent $0 < \alpha < 1$.

b) for any bounded measurable functions $\{f_k(t)\}$

$$\max_k \sup_{t \in I_k} |f_k(t)| \leq C \left(\sup_{(x_1,x_2) \in G} \left| \sum_{k=1}^{n} p_{m_k}(x_1, x_2) f_k(q_{m_k}(x_1, x_2)) \right| + \max_{k,j} |f_k(t_{k,j})| \right),$$

where C is a constant independent of the functions $\{f_k(t)\}$.

Let B be the direct sum of the spaces of bounded measurable functions on the intervals $\{I_k\}$:

$$B = \sum_{k=1}^{n} \oplus B(I_k).$$

The space B is a Banach space with the natural norm

$$\|\{f_k(t)\}\|_B = \max_k \sup_{t\in I_k} |f_k(t)|.$$

By $B(G)$ we denote the space of all bounded measurable functions $f(x_1, x_2)$ on G with uniform convergence.

Lemma 2. The linear operator $T : B \to B(G)$, acting according to the formula

$$T(\{f_k(t)\}) = f(x_1, x_2) = \sum_{k=1}^{n} p_{m_k}(x_1, x_2) f_k(q_{m_k}(x_1, x_2)),$$

maps bounded closed sets of the space B onto closed sets of the space $B(G)$.

The proof of Lemma 2 is based on the use of Lemma 1.

The following lemma from the theory of linear operators proved useful to us ([3]).

Lemma 3. Let B_1, B_2 be Banach spaces. If a linear operator $T : B_1 \to B_2$ maps bounded closed sets of the space B_1 onto closed sets of the space B_2, then its range is closed.

From Lemmas 1, 2, and 3 it follows that

Lemma 4. Let continuous functions $p_m(x_1, x_2)$ be fixed on the whole plane, and let functions $q_m(x_1, x_2)$, continuously differentiable on the whole plane, be fixed ($m = 1, 2, \dots, N$). Then in every domain D of the plane of the variables x_1, x_2 one can fix a closed subdomain $G \subset D$ such that the set of superpositions of the form

$$\sum_{m=1}^{N} p_m(x_1, x_2) f_m(q_m(x_1, x_2)), \tag{*}$$

where $\{f_m(t)\}$ are arbitrary bounded measurable functions, is closed in the space of all bounded measurable functions on G with the uniform metric.

Proof of Theorem 1. By a theorem of A. G. Vitushkin ([1]), the set of superpositions of the form (*) does not exhaust all continuous functions on G (in ([1]) the result is formulated under the assumption that the functions $\{f_m(t)\}$ are continuous, but it is proved even under the assumption that they are only bounded and measurable).

Consequently, by Lemma 4 the set of superpositions of the form (*) is nowhere dense in the space of all continuous functions on G, and hence we obtain that the set of these superpositions is nowhere dense also in the space of all continuous functions in the domain D. The theorem is proved.

By a method close to the method of proof of Lemma 1, one proves the following.

Lemma 5. For any continuous functions $p_m(x_1, x_2)$ and continuously differentiable functions $q_m(x_1, x_2)$ $(m = 1, 2, \dots, N)$, and any domain D, one can:

1) find a natural number ν;
2) fix a closed subdomain $G \subset D$;
3) indicate indices $1 \le m_1 < m_2 < \dots < m_n \le N$;
4) select, on the intervals I_k,

$$\Big\{ I_0 = \left[\min_G (x_1 + \nu x_2);\ \max_G (x_1 + \nu x_2)\right];$$

$$I_k = \left[\min_G q_{m_k}(x_1, x_2);\ \max_G q_{m_k}(x_1, x_2)\right], \quad k = 1, 2, \dots, n \Big\}$$

a finite set of points $t_{k,j} \in I_k$ $(k = 0, 1, 2, \dots, n;\ j = 1, 2, \dots, r_k)$ such that the following conditions are satisfied:

a) for all bounded measurable functions $\{\varphi_m(t)\}$ there exist bounded measurable functions $\{f_k(t)\}$ such that

$$\sum_{k=1}^{n} p_{m_k}(x_1, x_2)\, f_k(q_{m_k}(x_1, x_2)) \equiv \sum_{m=1}^{N} p_m(x_1, x_2)\, \varphi_m(q_m(x_1, x_2)) \quad \text{in } G;$$

b) for all bounded measurable functions $f_k(t)$ $(k = 0, 1, 2, \dots, n)$ satisfying the condition

$$f_0(x_1 + \nu x_2) = \sum_{k=1}^{n} p_{m_k}(x_1, x_2)\, f_k(q_{m_k}(x_1, x_2))$$

the inequality

$$\sup_{t \in I_0} |f_0(t)| \le C \left(\max_{k,j} |f_k(t_{k,j})| \right),$$

holds, where C is a constant independent of the functions $f_k(t)$ $(k = 0, 1, 2, \dots, n)$.

From Lemma 5 it follows:

Lemma 6. For any continuous functions $p_m(x_1, x_2)$ and continuously differentiable functions $q_m(x_1, x_2)$ $(m = 1, 2, \dots, N)$, and for any domain D, one can find a natural number ν and fix a closed subdomain $G \subset D$ such that the set of continuous functions $f_0(x_1 + \nu x_2)$ representable in the domain G by superpositions of the form (*) is a finite-dimensional linear space.

Proof of Theorem 2. Let l be the dimension of the finite-dimensional linear space of functions $f_0(x_1+\nu x_2)$ representable in the domain G by superpositions of the form (*) (see Lemma 6). Since the polynomials

$$(x_1+\nu x_2),\ (x_1+\nu x_2)^2,\ \dots,\ (x_1+\nu x_2)^{l+1}$$

are linearly independent, at least one of them, $Q(x_1,x_2) = (x_1+\nu x_2)^{\mu}$, is not equal in the domain G, and consequently also in the domain D, to any superposition of the form (*). The theorem is proved.

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References Cited

1. A. G. Vitushkin, *DAN*, **156**, No. 6 (1964).

2. A. N. Kolmogorov, *DAN*, **114**, No. 5 (1957).

3. N. Dunford, J. Schwartz, *Linear Operators, General Theory*, IL, 1962, p. 526.

Note: Figure translations are in progress. See original paper for figures.