On Complete Systems of Convergence

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Abstract

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A. M. Olevskii

On Complete Systems of Convergence

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An orthonormal system of functions $\{\varphi_n(x)\}$ in $L^2[0,1]$ is called a **system of convergence** (see ([2]), p. 357) if the series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \tag{1}$$

converges almost everywhere on $[0,1]$ as soon as the condition

$$\sum_{n=1}^{\infty} c_n^2 < \infty \tag{2}$$

is satisfied.

In other words, $\{\varphi_n(x)\}$ is a system of convergence if, for every function $f(x) \in L^2$, its Fourier series converges almost everywhere.

D. E. Menshov ([1]) was the first to establish the fundamentally important fact of the existence of orthonormal systems that are not systems of convergence.

A well-known example of a complete orthonormal system of convergence is the Haar system. For other classical complete systems of functions (the trigonometric system and the Walsh system), the question of whether they are systems of convergence remains unresolved to this day. For the trigonometric system this is the well-known Luzin problem.

It should be noted that the trigonometric system and the Walsh system consist of functions that are uniformly bounded in their entirety (i.e., $|\varphi_n(x)| < M$), whereas the Haar system, on the contrary, is constructed from high and narrow "peaks." In this connection the following problem arose: does there exist a complete orthonormal system of uniformly bounded functions that is a system of convergence? This question was formulated by P. L. Ul' yanov in survey articles ([4]), among other unsolved problems in the theory of orthogonal series.

The purpose of the present note is to prove the following theorem, which gives an affirmative answer to this question.

Theorem 1. *There exists a complete orthonormal system $\{\theta_n(x)\}$, consisting of uniformly bounded functions and being a system of convergence.*

We turn to the proof of this theorem. The construction of the system $\{\theta_n(x)\}$ we need will be carried out in several stages.

1. Let an arbitrary natural number k be fixed. Define the square matrix $A_k = \|a_{ij}^{(k)}\|$ of order 2^k as follows. Put

$$a_{1j}^{k} = 2^{-k/2} \quad (1 \le j \le 2^k). \tag{3}$$

Next, let $1 < i \le 2^k$. Then $i = 2^{s-1} + \nu$, where $1 \le \nu \le 2^{s-1}$. Put

$$a_{ij}^{(k)} = \begin{cases} 2^{(s-k-1)/2}, & (\nu-1)2^{k-s+1} + 1 \le j \le (2\nu-1)2^{k-s}, \\ -2^{(s-k-1)/2}, & (2\nu-1)2^{k-s} + 1 \le j \le \nu \cdot 2^{k-s+1}, \\ 0, & \text{for the remaining } j. \end{cases} \tag{4}$$

Thus the matrices A_k are defined for $k = 1, 2, \ldots$. We note that, in their structure, these matrices resemble the functions of the Haar system.

It is not hard to verify the following properties:

a) the matrix A_k is orthonormal for every k;

b)
$$\sum_{i=1}^{2^k} \left|a_{ij}^{(k)}\right| < C \qquad (1 \le j \le 2^k), \tag{5}$$

where, importantly, the constant C does not depend on k.

2. We now define an auxiliary system of functions $\{\psi_n(x)\}$. Each of these functions will be some polynomial in the Haar system $\{\chi_k^{(n)}(x)\}$ (see ([2]), p. 57).

Thus, set $\psi_0(x) = \chi_0^{(0)}(x)$, $\psi_1(x) = \chi_0^{(1)}(x)$. Next, let a number $n > 1$ be fixed. Then
$$n = 2^k + i - 1 \qquad (1 \le i \le 2^k). \tag{6}$$

Set
$$\psi_n(x) = \sum_{j=1}^{2^k} a_{ij}^{(k)} \chi_k^{(j)}(x). \tag{7}$$

Thus the system $\{\psi_n(x)\}$ $(n = 0, 1, 2, \ldots)$ is defined. It is not hard to verify the following properties:

a) $\{\psi_n(x)\}$ is a complete orthonormal system;

b)

$$\psi_{2^k-1}(x) = r_k(x) \qquad (k = 1, 2, ...) \tag{8}$$

(here $\{r_k(x)\}$ denotes the functions of the Rademacher system, see ([2]), p. 55);

c)

$$|\psi_n(x)| \leq \sqrt{n} \qquad (x \in [0,1];\ n = 0, 1, 2, ...). \tag{9}$$

These assertions follow immediately from (3), (7), and property a) of item 1.

Next denote by

$$\Delta_j^{(k)} = \left[\frac{j}{2^{k+1}}, \frac{j+1}{2^{k+1}}\right].$$

Let a natural number k be fixed. The union of the $\Delta_j^{(k)}$ over all even j from 0 to $2^{k+1}-2$ will be denoted by E_k. It is not hard to see that for any polynomial of the form

$$P(x) = \sum_{n=2^k}^{2^{k+1}-1} d_n \psi_n(x) \tag{10}$$

the equality

$$|P(x)| = \left|P\left(x + \frac{1}{2^{k+1}}\right)\right| \qquad (x \in E_k) \tag{11}$$

holds (here and below we neglect the endpoints of the intervals $\Delta_j^{(k)}$).

Now observe that if all the intervals $\Delta_j^{(k)}$ entering into E_k are shifted so that they adjoin one another, then we obtain a natural one-to-one mapping U_k of the set E_k onto the interval $[0, 1/2]$.

The following fact plays a central role in the constructions of this section:

$$\psi_n(x) = \chi_i(2U_{kx}) \qquad (2^k \leq n \leq 2^{k+1}-1;\ x \in E_k), \tag{12}$$

where i is determined from (6). (Here $\chi_i(x)$, $i = 1, 2, ...$, denote the functions of the Haar system, numbered consecutively.)

Using (11) and (12), as well as the properties of the Haar system, it is not hard to derive the following assertion: for any polynomial of the form (10) and any $2^k \leq m < 2^{k+1}$, the inequality

$$\left\|\sum_{n=2^k}^{m} d_n \psi_n(x)\right\|_{C[0,1]} \leq M\|P(x)\|_{C[0,1]},$$

holds, where M is an absolute constant.

Hence the following property follows:

c) every continuous function can be expanded in a uniformly convergent Fourier series with respect to the system $\{\psi_n(x)\}$.

From this, in turn, follows (see ([2]), pp. 182, 207) the property we need:

d) the system $\{\psi_n(x)\}$ is a convergence system.

3. Let us split the system $\{\psi_n(x)\}$ into three subsystems. Namely, as the first subsystem we take $\{\psi_{2^{2k}}(x)\}$, or, what is the same, by (8), $\{r_{2k+1}(x)\}$; as the second, $\{\psi_{2^{2k-1}}(x)\}$, i.e. $\{r_{2k}(x)\}$; finally, all the remaining functions of the system $\{\psi_n(x)\}$, in their former order, form the third subsystem (we denote it by $\{\psi'_n(x)\}$). We now combine the functions of the first and third subsystems as follows: put $\Phi_{2^k}(x) = \psi'_k(x)$; as $\Phi_n(x)$, $n \neq 2^k$, take the functions $\{r_{2l+1}\}$, without changing the order. It is not difficult to see that the system $\{\Phi_n(x)\}$ obtained in this way is orthonormal and becomes complete if the functions of the second subsystem are added to it.

Further, from (8) and (9) it is obvious that

$$|\Phi_n(x)| \leq \sqrt{n} \quad (n = 2^k); \qquad |\Phi_n(x)| \leq 1 \quad (n \neq 2^k); \quad x \in [0,1]. \tag{13}$$

The system $\{\Phi_n(x)\}$ naturally splits into "blocks" of 2^k functions in the k-th block; therefore we can number this system, similarly to the Haar system, with a double subscript $\Phi_k^{(m)}$ $(1 \leq m \leq 2^k,\ k = 0,1,2,...)$. In this notation $\Phi_k^{(1)} = \psi'_k(x)$.

4. Put $\varphi_1(x) = \Phi_1^{(0)}(x)$ and, further,

$$\varphi_n(x) = \sum_{m=1}^{2^k} a_{mi}^{(k)} \Phi_k^{(m)}(x), \tag{14}$$

where i and k are determined from (6). We now define the system $\{\theta_n(x)\}$. Namely, put $\theta_{2^k+k}(x) = r_{2k+2}(x)$. As $\theta_n(x)$, $n \neq 2^k + k$, take the functions of the system $\{\varphi_n(x)\}$ in their original order. Thus the system $\{\theta_n(x)\}$ $(n = 1,2,...)$ is completely defined. It is easy to see that it is a complete orthonormal system. Further, using (3), (5), and (13), it is not difficult to show that it consists of functions bounded in their totality. Finally, let us show that it is a convergence system. It suffices to prove this fact for the system $\{\varphi_n\}$. Denote

$$P_n(x) = a_{1i}^{(k)} r_{2k+2}(x) + \sum_{m=2}^{2^k} a_{mi}^{(k)} \Phi_k^{(m)}(x).$$

Consider an arbitrary series (1) with coefficients satisfying condition (2). We have:

$$\sum_{n=2}^{\infty} c_n \varphi_n(x) = \sum_{k=1}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} c_n \sum_{m=1}^{2^k} a_{mi}^{(k)} \Phi_k^{(m)}(x) =$$

$$= \sum_{k=1}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} c_n \left[P_n(x) - a_{1i}^{(k)} r_{2k+2}(x) + a_{1i}^{(k)} \psi'_k(x)\right]. \qquad (15)$$

Thus the series (1) is decomposed into the sum of three series. The convergence of the first of them can be established by the method of Erdős (see ([3]), p. 706), if one takes into account that $\{P_n(x)\}$ is an orthonormal system of functions, each of which is a polynomial in the Rademacher system. In this case, instead of the lemma

Sidon used in (3), we shall have to refer to Khintchine' s inequality (see (2), p. 153) for the case $p = 4$.

Further, evidently,

$$\sum_{n=2^k}^{2^{k+1}-1} \left|c_n a_{1i}^{(k)}\right| \leq \left(\sum_{n=2^k}^{2^{k+1}-1} c_n^2\right)^{1/2} = b_k; \qquad \sum_{k=1}^{\infty} b_k^2 < \infty.$$

Taking into account that $\{\psi'_k\}$ and $\{r_k\}$ are systems of convergence, we obtain that the partial sums with numbers $n_k = 2^{k+1} - 1$ of the second and third of the series (15) converge almost everywhere. Finally,

$$\delta_k(x) = \max_{2^k \leq m \leq 2^{k+1}-1} \left|\sum_{n=2^k}^{m} c_n a_{1i}^{(k)} \left[\psi'_k(x) - r_{2k+2}(x)\right]\right| \leq b_k \left(|\psi'_k(x)| + |r_{2k+2}(x)|\right).$$

Squaring and integrating, we obtain that $\lim_{k\to\infty} \delta_k(x) = 0$ almost everywhere.

Thus the convergence of the series (1) has been established. It follows that the system $\{\varphi_n(x)\}$, and together with it also $\{\theta_n(x)\}$, are systems of convergence. Thus the proof of the theorem is complete.

Let us note that it is clear from the proof that the system $\{\theta_n\}$ constructed by us is complete not only in $L^2[0,1]$, but also in $L[0,1]$. It should also be noted that the system $\{\theta_n(x)\}$ has some other curious properties. Thus, for example, with its help the following theorem can be proved.

Theorem 2. *There exist a complete in $L[0,1]$, orthonormal, uniformly bounded system $\{\varphi_n(x)\}$ and a function $f(x)$, belonging to all classes L^p for $1 \leq p < 2$, such that the partial sums $s_{n_k}(x)$ of the Fourier series (1) of this function diverge almost everywhere on $[0,1]$, whatever the sequence $n_k \uparrow \infty$ may be.*

Such a result was first obtained by Marcinkiewicz, but only for $p < 6/5$ (see (2), p. 358). It is evident that our theorem is final in this respect–the condition $p < 2$ cannot be strengthened. It is useful to note that the system $\{\varphi_n(x)\}$ from Theorem 2 is obtained from the system $\{\theta_n(x)\}$ by means of a certain permutation of the elements, which, however, does not destroy its basic property —being a system of convergence. Thus we may draw the following conclusion: for complete and even uniformly bounded systems, convergence almost everywhere of Fourier series from L^2 does not imply results concerning convergence of series from L^p for $p < 2$. In this connection it is natural to raise the question of the existence of a complete orthonormal uniformly bounded system for which every Fourier–Lebesgue series converges almost everywhere. As is known, the trigonometric system does not have this property (A. N. Kolmogorov's example).

In conclusion let us note that the construction used in building the system $\{\theta_n(x)\}$ may prove useful also in the investigation of some other questions. Thus, for example, with the aid of the matrices A_k of item 1 one can without difficulty construct an example of a lacunary system giving a negative answer to the question posed in the book (2) (p. 297).

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Moscow Institute
of Electronic Machine Building

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Note: Figure translations are in progress. See original paper for figures.