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N. Ya. KRUPNIK

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Abstract

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N. Ya. KRUPNIK

ON MULTIDIMENSIONAL SINGULAR OPERATORS IN SPACES OF TEST AND GENERALIZED FUNCTIONS

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In the monograph of S. G. Mikhlin ([1]), along with a number of other problems, the problem is posed of studying multidimensional singular operators in spaces of generalized functions. In this direction a number of results are already available. In ([7]) the continuity of singular operators* (with symbol independent of the pole) in the space D'_{L_p} ([9]) is proved, and the composition formulas for singular integrals are extended to the case in which their densities are generalized functions. In ([5]) the boundedness of singular operators in the spaces of S. L. Sobolev $W_p^{(l)}$ $(1 < p < \infty)$, as well as in the spaces conjugate to them, is proved. In addition, it is shown that the operator $A_1A_2 - A_3$, where A_3 is the singular operator with symbol equal to the product of the symbols of the operators A_1 and A_2, acts from $W_p^{(l)}$ into $W_p^{(l+1)}$. In ([6]) a number of estimates are derived for singular integral operators whose kernels are generalized functions. In the present paper the space $\mathfrak{M}$ of test functions and $\mathfrak{M}^*$ of generalized functions ($\mathfrak{M}^* \supset D'_{L_p}$ for $p < \infty$) are introduced, and the question of the normal solvability and the index of singular operators (as well as of matrix singular operators) in the spaces $\mathfrak{M}$, D_{L_p}, $W_p^{(l)}$, $W_p^{(l)*}$, D'_{L_p}, and $\mathfrak{M}^*$ is investigated.

Moreover, the impossibility is proved (in a certain sense) of extending singular operators to spaces broader than $\mathfrak{M}^*$.

Let us agree on the following notation: E_m is m-dimensional Euclidean space; $q = (q_1, \dots, q_m)$, where the q_i are nonnegative integer coordinates, $|q| = q_1 + \dots + q_m$; $D^q\varphi = \partial^{|q|}\varphi/\partial x_1^{q_1} \cdots \partial x_m^{q_m}$; C is one of the singular operators ([1]) A, B, where

$$A\varphi = a(x)\varphi(x) + \int_{\dot{E}_m} \frac{f(x,\theta)}{r^m}\varphi(y)\,dy, \qquad B\varphi = \bar{a}(x)\varphi(x) + \int_{\dot{E}_m} \frac{\bar{f}(y,-\theta)}{r^m}\varphi(y)\,dy;$$

$f^{(q)}(x,\theta)$ is the derivative $D^q f(x,\theta)$, computed under the assumption that θ

does not depend on x; $W_p^{(l)}(E_m)$ is the S. L. Sobolev space with norm

$$\|\varphi\| = \sum_{|q|\le l} \|D^q\varphi\|_{L_p(E_m)},$$

if $l \ge 0$, and

$$W_p^{(l)}(E_m) = [W_s^{(-l)}(E_m)]^*$$

$(s^{-1}+p^{-1}=1)$, if $l<0$; $M=\bigcup_\alpha A_\alpha$ $(0<\alpha<1)$ (see ([1]), p. 64). We shall say that $\varphi \in M^{(n)}$ if $D^q\varphi \in M$ for all $|q| \le n$. Everywhere, unless otherwise stated, we shall assume that the necessary condition for the existence of the singular integrals ([1]) is fulfilled,

$$\int_S f(x,\theta)\,dS = 0.$$

* In the present paper, the notation and terminology of S. G. Mikhlin ([1]) are used for the most part.

1. Let $M_p = (1+|x|)^{m-\frac{1}{p+1}}$; denote by $\mathfrak{M}$ the set of complex infinitely differentiable functions $\varphi(x)$ on E_m for which the products $M_p|D^q\varphi|$, for $|q| \le p$ $(p=0,1,...)$, are bounded on all of E_m. The topology introduced by means of the countable collection of norms

$$\|\varphi\|_p = \sup_{|q|\le p} M_p|D^q\varphi| \quad (p=0,1,...)$$

turns $\mathfrak{M}$ into a complete countably normed space of type $K\{M_p\}$ ([2]).

Suppose that for each fixed $z \ne 0$ the functions $a(x)$ and $f(x,z/|z|)$ are infinitely differentiable with respect to x and

$$|D^q a(x)| \le N_q, \qquad |f^{(q)}(x,\theta)| \le N'_q \quad (N_q, N'_q = \text{const}); \tag{1}$$

then one can show that the operators A and B are continuous in $\mathfrak{M}$, whence it follows that A^* and B^* are continuous in $\mathfrak{M}^*$. But on the linear set $L_p(E_m)$, dense in $\mathfrak{M}^*$, $A^*u = Bu$ and $B^*u = Au$; this makes it possible to define in $\mathfrak{M}^*$ the operators A and B, respectively, by the equalities $(AF,\varphi)=(F,B\varphi)$, $(BF,\varphi)=(F,A\varphi)$, where $\varphi\in\mathfrak{M}$, $F\in\mathfrak{M}^*$.

2. We formulate three lemmas which we shall use in the proof of Theorems 1-4.

Lemma 1. Suppose C_3 is the operator with symbol equal to the product $\Phi_1(x,\theta)\Phi_2(x,\theta)$ of the symbols of the operators C_1 and C_2. If for every $|q| \le 2$ the functions $\Phi_1^{(q)}(x,\theta)$ and $\Phi_2^{(q)}(x,\theta)$, and their derivatives up to order $m+[m/2]+1$ with respect to the Cartesian coordinates of the points θ, are continuous on $\Sigma \times S^*$, then for any $1<p<\infty$

$$T(M) \equiv (C_1C_2 - C_3)(M) \subset W_p^{(1)}(E_m)^{**}.$$

We introduce the following notation. We say that $\Phi(x,\theta) \in G_n$ if:
1) for every $|q| \leq 2$ the functions $\Phi^{(q)}(x,\theta)$ and their derivatives up to order $m + [m/2] + 1$ with respect to the Cartesian coordinates of the points θ are continuous on $\Sigma \times S$;
2) for all $|q| \leq n+1$, $\Phi^{(q)}(x,\theta)$ are continuous on $\Sigma \times S$, and 3) $\Phi^{(q)}(x,\theta)\hat{} \in \hat{W}_p^{(l)}(S)$, where $l \geq m+[m/2]+2$, if $|q| \leq n$, and $l \geq m+[m/2]+1$, if $|q| = n+1$.

Lemma 2. If the symbol of the operator C, $\Phi(x,\theta) \in G_n$, $\inf|\Phi| > 0$, $\varphi \in L_2(E_m)$, $\psi \in M^{(n)}$, and $C\varphi = \psi$, then $\varphi \in M^{(n)}$.

Lemma 3. Suppose $\Phi(x,\theta) \in G_n$ for some n and $\inf|\Phi| > 0$. If $C\varphi = 0$ and $\varphi \in L_{p_0}$ for some $1 < p_0 < \infty$, then $\varphi \in L_p$ for every p $(1 < p < \infty)$.

Theorem 1. If $\Phi(x,\theta) \in G_n$ and $\inf|\Phi| > 0$, then the operator C is normally solvable in $W_p^{(l)}(E_m)$ $(1 < p < \infty,\ |l| \leq n)$, and its index is equal to zero.

We carry out the proof according to the following scheme: we show that the operators $C'C - I$ and $CC' - I$, where C' is the singular operator with symbol $\Phi^{-1}(x,\theta)$, and I is the identity operator, are completely continuous in $W_p^{(l)}(E_m)$, whence the normal solvability and finiteness of the index of the operator C follow ([3]). Then, using Theorem 4 of the work ([4]), we prove that the index of the operator C is equal to zero.

Corollary 1. Suppose the conditions of Theorem 2 are fulfilled and $|l| \leq n$. If $\psi \in M^{(l)}$ $(\psi \in W_p^{(l)})$, $\varphi \in W_s^{(-n)}$ $(1 < s, p < \infty)$, and $C\varphi = \psi$, then $\varphi \in M^{(l)}$ $(\varphi \in W_p^{(l)})$.

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* Here S is the unit sphere in E_m, and Σ is the unit sphere in E_{m+1}, into which E_m is mapped under the stereographic transformation ([1]).

** Under the condition that the symbols are infinitely differentiable with respect to the Cartesian coordinates of the points θ on S, in ([5]) it is proved that $T(L_p) \subset W_p^{(1)}$.

Corollary 2. If $\Phi(x,\theta) \in G_n$ for all n and $\inf|\Phi| > 0$, then the operator C is normally solvable in the L. Schwartz spaces D'_{L_p} and D'_{L_p} ([9]), and its index is equal to zero.

Theorem 2. If $\Phi(x,\theta) \in G_n$ for all n and $\inf|\Phi| > 0$, then the operator C is normally solvable in the spaces $\mathfrak{M}$ and $\mathfrak{M}^*$, and its index in these spaces is equal to zero.

We carry out the proof according to the following scheme: we represent the operator C in the form of a sum $C = C_1 + K$, where C_1 is invertible in $L_2(E_m)$, and K is a finite-dimensional operator. Then we show that the operator K acts in $\mathfrak{M}$ and that C_1 is invertible in $\mathfrak{M}$; consequently ([8]), the operator C is normally solvable in $\mathfrak{M}$ and its index is equal to zero. The corresponding assertions in the space $\mathfrak{M}^*$ are proved by passing to adjoint operators.

3. Denote by $\mathfrak{M}^k$ the set of vector-functions $\varphi = (\varphi_1, \dots, \varphi_k)$, where $\varphi_i \in \mathfrak{M}$. If in $\mathfrak{M}^k$ we introduce the countable system of norms

$$\|\varphi\|_{kp} = \left(\sum_{i=1}^{k} \|\varphi_i\|_p^2 \right)^{1/2},$$

where $\|\varphi_i\|_p$ is the p-th norm of the function $\varphi_i \in \mathfrak{M}$, then $\mathfrak{M}^k$ becomes a complete countably normed space. Consider the system

$$\sum_{j=1}^{k} C_{ij}\varphi_j = \psi_i \qquad (i = 1, 2, \dots, k), \tag{2}$$

where C_{ij} are operators of type A and B. Let C be the matrix singular operator ([1]) corresponding to the system (2).

Theorem 3. If the symbols of the operators C_{ij}, $\Phi_{ij} \in G_n$, and $\inf|\det\Phi| > 0$, then for $1 < p < \infty$ and $|l| \le n$ the operator C is normally solvable in $W_p^{(l)k}$ and has a finite index, moreover

$$\operatorname{ind}_{W_p^{(l)k}}(C) = \operatorname{Ind}_{L_2^k}(C).$$

Theorem 4. If $\Phi_{ij} \in G_n$ for all n and $\inf|\det\Phi| > 0$, then the operator C is normally solvable in $\mathfrak{M}^k$ and $(\mathfrak{M}^k)^*$, and

$$\operatorname{Ind}_{\mathfrak{M}^k}(C) = \operatorname{Ind}_{(\mathfrak{M}^k)^*}(C) = \operatorname{Ind}_{L_2^k}(C). \tag{3}$$

We carry out the proof according to the following scheme. Let S be a matrix singular operator with symbolic matrix $\Phi^{-1}(x, \theta)$. Then the operators SC and CS are normally solvable in L_2^k, and their indices are equal to zero. By the same scheme as in theorem 2, we prove that SC and CS are normally solvable in $\mathfrak{M}^k$ and $(\mathfrak{M}^k)^*$, and their indices are equal to zero. Then ([8]) the operators S and C are normally solvable in $\mathfrak{M}^k$ and $(\mathfrak{M}^k)^*$, and have finite indices. The validity of equality (3) follows from the fact that

$$\operatorname{Ind}_{\mathfrak{M}^k}(C) \le \operatorname{Ind}_{L_2^k}(C), \qquad \operatorname{Ind}_{\mathfrak{M}^k}(S) = \operatorname{Ind}_{L_2^k}(S),$$
$$\operatorname{Ind}_{\mathfrak{M}^k}(C) + \operatorname{Ind}_{\mathfrak{M}^k}(S) = \operatorname{Ind}_{L_2^k}(C) + \operatorname{Ind}_{L_2^k}(S).$$

Remark. In the monograph of S. G. Mikhlin ([1]) several sufficient conditions are given for the index of the system (2) in $L_p^k(E_m)$ to be equal to zero (see Theorems 2.40–5.40). From theorem 3 ([4]) it follows that each of the conditions of S. G. Mikhlin gives rise to a corresponding sufficient condition for the index of the system (2) in $W_p^{(l)k}$ $(\mathfrak{M}^k, (\mathfrak{M}^k)^*)$ to be equal to zero.

4. Since the space $\mathfrak{M}$ contains all finite infinitely differentiable functions, the values of a regular functional (F, φ) on basic functions uniquely (up to values on a set of measure zero) determine the corresponding locally integrable function $F(x)$ ([2]). The space $\mathfrak{M}^*$ contains all locally integrable func-

tions for which the products $|F(x)|(1+|x|)^{-m+\varepsilon}$ (for some $\varepsilon > 0$) are summable on E_m.

Let Q be a linear topological space such that Q^* contains all locally integrable functions for which the products $|F(x)|(1+|x|)^k$, where $k < m$ is fixed, are summable on E_m.

Consider the following problem. Is it possible, by the same method as above, to extend singular operators to the space Q^* (which is broader than $\mathfrak{M}^*$)? In order to extend the operator A to Q^*, one must find a space P such that $B(P) \subset Q$ and define $(AF, \varphi) = (F, B\varphi)$, where $F \in Q^*$, $\varphi \in P$. Suppose that the symbol of the operator C does not depend on the pole. A negative answer to the problem posed above is given by

Theorem 5. *If there exists a linear topological space $P \subset Q$, sufficiently rich in functions, such that $C(\dot{P}) \subset Q$, then the characteristic of the operator C satisfies $f(\theta) \equiv 0$.*

Kishinev State
University

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* The space P is said to be sufficiently rich in functions if from the equality $(F, \varphi) = 0$, where F is locally integrable and φ ranges over all P, it follows that $F(x) = 0$ almost everywhere.

Note: Figure translations are in progress. See original paper for figures.