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Abstract

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ON LIMIT CYCLES OF THE EQUATIONS OF MOTION OF A RIGID BODY AND OF GALERKIN EQUATIONS OF HYDRODYNAMICS

(Presented by Academician L. S. Pontryagin on 23 III 1964)

This note investigates the birth of limit cycles from stationary solutions of the system of equations

$$\begin{aligned} \dot{x} &= Ayz + F_1 - \nu\lambda_1 x, \\ \dot{y} &= Bxz + F_2 - \nu\lambda_2 y, \\ \dot{z} &= Cxy + F_3 - \nu\lambda_3 z, \end{aligned} \tag{1}$$

where

$$A + B + C = 0^*. \tag{2}$$

For $\nu = 0$, equation (1) becomes Euler's equations for the motion of a rigid body fixed at its center of gravity. Here x, y, z are the vector of kinetic moment relative to the body; the coefficients A, B, C are expressed in terms of the moments of inertia; F_1, F_2, F_3 are the components of the moment of the external forces. The dissipative terms $-\nu\lambda_1 x$, $-\nu\lambda_2 y$, $-\nu\lambda_3 z$ have the meaning of friction. Condition (2) expresses the law of conservation of energy in the absence of dissipative terms and external forces.

V. I. Yudovich ([1]) proposed considering system (1) as the simplest model of the Navier–Stokes equations. Indeed, the Galerkin system of ordinary differential equations for the Fourier coefficients has the structure of system (1) (see also ([2])). Here ν denotes viscosity; F_1, F_2, F_3 are the Fourier coefficients of the body forces.

§ 1. We shall study the dependence of the solutions of system (1) on ν^{**} for fixed F_1, F_2, F_3 and

$$A > 0, \qquad B < 0, \qquad C > 0; \qquad \lambda_1,\ \lambda_2,\ \lambda_3 > 0. \tag{3}$$

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For $\nu = \infty$, system (1) has the unique "laminar" equilibrium position 0. All solutions as $t \to +\infty$ tend to it. As ν decreases, the coordinates of the equilibrium position $x(\nu), y(\nu), z(\nu)$ change continuously. For large ν the equilibrium remains stable.

The purpose of the present note is to indicate sufficient conditions for loss of stability of a stationary solution and the birth from it of a limit cycle as ν passes through the critical value ν_{cr}.

Theorem 1. Let the coefficients $A, B, C, F_1, F_2, F_3, \lambda_1, \lambda_2, \lambda_3$ satisfy the inequality

$$H = \frac{\lambda_2 + \lambda_3}{\lambda_1^2} BCF_1^2 + \frac{\lambda_1 + \lambda_3}{\lambda_2^2} ACF_2^2 + \frac{\lambda_1 + \lambda_2}{\lambda_3^2} ABF_3^2 > 0. \tag{4}$$

$*$ In interpreting system (1) as equations of motion of a rigid body, one must bear in mind that between the coefficients A, B, and C, in addition to condition (2), there are also relations of the type of inequalities. However, Theorems 1 and 2 of this paper remain valid even when these restrictions are taken into account (the regions described in Theorems 1 and 2 must be intersected with the added inequalities).

$**$ In hydrodynamics, the Reynolds number Re is chosen as the parameter; its magnitude is inversely proportional to the viscosity ν.

Then the equilibrium position $x(\nu), y(\nu), z(\nu)$ is stable only for $\nu > \nu_{\text{cr}}$. For $\nu = \nu_{\text{cr}}$, a pair of eigenvalues with nonzero imaginary parts passes from the left half-plane into the right half-plane. The third eigenvalue remains negative.

Inequality (4), under condition (3), singles out in the space of the coefficients F_1, F_2, F_3 a domain bounded by the surface of the hyperboloid $H = \text{const}$.

Theorem 2. *When ν passes through the critical value ν_{cr}, two cases are possible:*

1) from the equilibrium position a stable limit cycle is born; the radius of the cycle for $\nu = \nu_{\text{cr}} - \varepsilon$ is of order $\sqrt{\varepsilon}$;

2) an unstable limit cycle is drawn into the equilibrium position; the radius of the cycle for $\nu = \nu_{\text{cr}} + \varepsilon$ is of order $\sqrt{\varepsilon}$.

The first case occurs for $g < 0$, the second for $g > 0$, where the quantity g is a function of all the coefficients of system (1) (see (3)).

§ 2. Let us expand the solution of the nonstationary Navier–Stokes equations for the spatially periodic problem in a Fourier series. The Galerkin system of N ordinary differential equations for the Fourier coefficients can be written (see ([1], [2], [10], [11])) in the form*

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$$\dot{x}_k = \sum_{l,m=1}^{N} A_{klm} x_{lx} m + F_k + \nu \lambda_k x_k \tag{5}$$

$$(k, l, m = 1, \dots, N).$$

We shall assume that

$$\sum_{k,l,m=1}^{N} A_{klm} x_{k x_{lx}} m \equiv 0 \tag{6}$$

(this is the expression of the law of conservation of energy of an ideal fluid in the absence of body forces and viscosity,

$$\frac{d}{dt} \sum_{k=1}^{N} \frac{x_k^2}{2} = 0$$

).

Again we shall follow the stationary solution $x_k(\nu)$, which tends to 0 as $\nu = \infty$.

Theorem 3. *The assertions of Theorems 1 and 2 remain valid for the N-dimensional system (5) with condition (6). Inequality (4) in the space of the coefficients $F_1, \dots, F_N$ (for fixed $A_{klm}, \lambda_1, \dots, \lambda_N$) singles out a domain analogous to that described in Theorem 1.*

The quantity g is a function of all the coefficients of the system, not identically equal to zero in any subdomain of the space of coefficients $A_{klm}, \lambda_1, \dots, \lambda_N, F_k$ satisfying conditions (6) and $H > 0$ (see (3)). In this space the function g changes sign.

Thus, both cases indicated in Theorem 2 are realized. In case 1), the stability of the stationary solution passes to the cycle. Case 2) means that the physically observed stability can be lost earlier than the mathematical one.**

The mathematical results presented correspond to the picture of loss of stability indicated in § 27 (the onset of turbulence) of the book ([4]). The mathematical problem of finding the conditions for the birth of limit cycles in this situation was posed by A. N. Kolmogorov ([5]).

The passage of eigenvalues through zero (the "principle of loss of stability," §§ 2.1, 4.2, 7.3, 7.4 of the book ([6])) is not obligatory.

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* All the results formulated in the note are also valid for the Galerkin system corresponding to the case of a vessel with immobile walls, when the solution is

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sought in the form of a series in eigenfunctions of the linear problem (self-adjoint and positive-definite).

** That is, the domain of attraction of the equilibrium position becomes small as ν decreases.

§ 3. Let us return to system (1) and consider $\nu \to 0$. Suppose, for definiteness, that the conditions $F_1, F_2, F_3 > 0$ and (3) are satisfied. For $\nu = 0$ system (1) has two unstable equilibrium positions. The complex eigenvalues of one of them have positive real part. Fix the coefficients A, B, C, F_k, λ_k, and follow the indicated equilibrium position $x(\nu), y(\nu), z(\nu)$ as ν increases.

In this case limit cycles may also be born, since, as ν increases, the eigenvalues pass into the left half-plane.

Theorems 1 and 2 are valid here as well. The form of the functions H and g, naturally, will be different, since these expressions depend on the point in whose neighborhood the right-hand sides are expanded. However, in this case too the conditions $H > 0$ and $g < 0$ single out, in the space of coefficients of system (1), a region analogous to that described above. In exactly the same way, in the multidimensional system (5) with condition (6), as the "viscosity" ν decreases, in addition to the stationary solution considered in § 2, secondary stationary solutions arise. The mathematical results of this section show that stable periodic motions may arise away from the principal stationary solution observed for all ν.

§ 4. The proof of Theorem 1 is cumbersome. We fix the coefficients A_{klm}, λ_k and a sufficiently small ν_{cr}. Then the coefficients $F_1, \dots, F_N$ are chosen so that a pair of eigenvalues is purely imaginary at $\nu = \nu_{\text{cr}}$. This proves possible under condition (4). Thus the set swept out by the admissible $F_1, \dots, F_N$ for all ν_{cr} is bounded by a conical surface.

The proof of Theorem 2 is based on the general theorem of the paper ([3]). To verify condition 3 of the paper ([3]), one must reduce systems (1), (5) to the form (1_ε) of the paper ([3]). For this purpose one must choose the coordinates x_i so that the origin of coordinates is at the equilibrium position. The coordinate axes x_1, x_2 must be chosen in the plane corresponding to the pair of critical eigenvalues. For large ν this plane will be close to the plane $x_1 x_2$ (see (3)). The choice of the remaining axes in the invariant complement is immaterial; therefore the results of the paper ([3]) are easy to use in the multidimensional case.

In order that system (1) take the form (1_ε) of the paper ([3]), it is also necessary, by a change of variables, to eliminate from the first two equations the terms containing the products $x_1 x_i, x_2 x_i$ $(i = 3, \dots, N)$. Such a change is always possible on the basis of Theorem 2 ([3]) on the existence of a two-dimensional invariant manifold on which the limit cycle is born. The result of Theorem 2 of the paper ([3]) greatly simplifies the proof of Theorem 3 of the present paper for large N.

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§ 5. From the proofs of Theorems 1, 2, 3 it follows that, for any $A_{klm}, \lambda_1, \ldots, \lambda_N$, one can choose the free terms $F_1, \ldots, F_N$ so that system (1) ((5)) has a limit cycle.

We shall interpret system (1) as the equations of motion of a rigid body fixed at its center of gravity, under the action of a moment F_1, F_2, F_3 that is constant relative to the body and damping forces $-\nu\lambda_1 x, -\nu\lambda_2 y, -\nu\lambda_3 z$. It is usually assumed ([7])* that damping of such a system leads to a stationary rotation of the body about an axis close to one of the axes of inertia. Our result shows that, for any body ($A, B, C, \lambda_1, \lambda_2, \lambda_3$ arbitrary), one can choose an external moment $F = \{F_1, F_2, F_3\}$ so that the axis of rotation performs periodic oscillations in the body.

We shall now regard system (5) as the Galerkin system for the Navier–Stokes equations. Then our result means that the body force $F = \{F_1, \ldots, F_N\}$ can be chosen so that, when stability is lost–

* See the bibliography of the book ([7]).

eigenvalues pass into the right half-plane not through zero.

A mathematically rigorous investigation of the transition of eigenvalues into the right half-plane has been carried out only in the example of L. D. Meshalkin and Ya. G. Sinai (^8); see also (^9). In these examples the eigenvalues pass into the right half-plane through zero.

In the model proposed by A. N. Kolmogorov (^5) and considered by L. D. Meshalkin and Ya. G. Sinai (^8), the vector of body forces F has Fourier components $\{1, 0, \ldots, 0, \ldots\}$. By changing in the model (^8) only the vector of body forces, one can obtain a limit cycle in the corresponding Galerkin system (5) for any N; for example, take F to be $\{0, 1, 0, \ldots, 0, \ldots\}$.

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