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Abstract

Full Text

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ELLIPTIC PROBLEMS WITH NONHOMOGENEOUS BOUNDARY CONDITIONS AND LOCAL INCREASE OF SMOOTHNESS UP TO THE BOUNDARY OF GENERALIZED SOLUTIONS

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For the case of elliptic equations and general homogeneous boundary conditions, in ⁽¹⁾ a theorem on the complete set of homeomorphisms was established. In ^(1,2) this theorem was applied to the local increase of smoothness up to the boundary of generalized solutions of elliptic equations. We note that the theorem on increase of smoothness from ⁽²⁾ contained certain unnatural restrictions.

In the present note a theorem is established on the complete set of homeomorphisms for elliptic equations with nonhomogeneous boundary conditions. The homeomorphisms are established between such spaces as are adapted to questions of increasing the smoothness of generalized solutions. With the aid of the established theorem, an assertion on the local increase of smoothness up to the boundary of generalized solutions is proved; from it it follows, in particular, that the corresponding results of the papers ^(3,4) on the smoothness of the Green's function and of the spectral function of elliptic operators are valid without the additional restrictions a), b) on the boundary differential expressions.

We note that homeomorphisms for other spaces in the case of nonhomogeneous conditions were studied in the papers of Lions and Magenes ⁽⁵⁾. Some questions concerning increase of smoothness are considered in the new book of Hörmander ⁽¹¹⁾.

1°. Let G be a bounded domain of the space E_n , and let Γ be its boundary. In \overline{G} there is given a properly elliptic ⁽⁶⁾ expression \mathcal{L} of order $2m$ with complex coefficients,

$$\mathcal{L} = \sum_{|\mu| \leq 2m} a_\mu(x) D^\mu \quad (1)$$

$$\left(\mu = (\mu_1, \dots, \mu_n); |\mu| = \mu_1 + \dots + \mu_n; D^\mu = D_1^{\mu_1} \dots D_n^{\mu_n}; D_k = \frac{1}{i} \frac{\partial}{\partial x_k} \right).$$

On Γ there are given m differential expressions

$$B_j = \sum_{|\mu| \leq m_j} b_{j\mu}(x) D^\mu, \quad m_j \leq 2m - 1; \quad j = 1, \dots, m. \quad (2)$$

We assume that the boundary expressions (2) cover \mathcal{L} in the sense of (6). Let $W_2^k(G)$ ($k \geq 0$ an integer) be the space of S. L. Sobolev, $W_2^{-k}(G)$ the space with negative norm (7), constructed from the null $L_2(G)$ and positive $W_2^k(G)$; we shall denote the corresponding norms by $\| \cdot \|_k$, $\| \cdot \|_{-k}$, $W_2^{k-\frac{1}{2}}(\Gamma)$ ($k > 0$ an integer) the space of functions φ defined on the boundary Γ of the domain G and which are the values on Γ of functions $u \in W_2^k(G)$. The norm in this space may be defined, for example, as follows:

$$\|\varphi, \Gamma\|_{k-1/2} = \|\varphi\|_{W_2^{k-1/2}(\Gamma)} = \inf \|u\|_k,$$

where the infimum is taken over all $u \in W_2^k(G)$ equal to φ on Γ . As is known, with the aid of the Fourier transform one can define an equivalent norm and the corresponding scalar product, relative to which $W_2^{k-1/2}(\Gamma)$ forms a Hilbert space.

space. We shall denote the scalar products in $L_2(G)$ and $L_2(\Gamma)$ by $(\cdot, \cdot)_0$ and $\langle \cdot, \cdot \rangle_0$, respectively. We shall also consider orthogonal sums of spaces

$$W_2^s(G) \oplus \sum_{j=1}^m W_2^{k_j+1/2}(\Gamma) \quad (s, k_j \text{ integers}),$$

in particular, we denote

$$K_s = W_2^s(G) \oplus \sum_{j=1}^m W_2^{2m-m_j-1/2+s}(\Gamma).$$

If

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_m) \in W_2^{-s}(G) \oplus \sum_{j=1}^m W_2^{-(k_j+1/2)}(\Gamma),$$

and

$$v = (v_0, v_1, \dots, v_m) \in W_2^s(G) \oplus \sum_{j=1}^m W_2^{k_j+1/2}(\Gamma),$$

then we denote

$$[\alpha, v] = (\alpha_0, v_0) + \sum_{j=1}^m \langle \alpha_j, v_j \rangle_0.$$

2°. Consider the operator \mathfrak{A} :

$$\mathfrak{D}(\mathfrak{A}) = W_2^{2m}(G); \quad \mathfrak{A}u = (\mathcal{L}u, B_1u, \dots, B_mu) \in K_0.$$

The operator \mathfrak{A} acts continuously from $W_2^{2m}(G)$ into K_0 . Let N be the kernel of this operator: $u \in N$ if $\mathfrak{A}u = 0$. As is known ⁽⁸⁻¹⁰⁾, under the corresponding smoothness conditions the kernel of the operator \mathfrak{A} is finite-dimensional, the range $\mathfrak{R}(\mathfrak{A})$ is closed in K_0 , and has finite codimension.

Suppose now that the boundary expressions (2) are normal ⁽⁶⁾. Then Green's formula ^(5,6) is valid:

$$(\mathcal{L}u, v)_0 + \sum_{j=1}^m \langle B_{ju}, C'_{jv} \rangle_0 = (u, \mathcal{L}^+v)_0 + \sum_{j=1}^m \langle C_{ju}, B'_{jv} \rangle_0$$

$$(u, v \in W_2^{2m}(G)). \quad (3)$$

Here \mathcal{L}^+ is the expression formally adjoint to \mathcal{L} ;

$$B'_j = \sum_{|\mu| \leq m'_j} b'_{j\mu}(x) D^\mu \quad (j = 1, \dots, m) \quad (4)$$

is a system of normal differential expressions covering \mathcal{L}^+ ; $\{C_j\}$, $\{C'_j\}$, ($j = 1, \dots, m$), are differential expressions of type (2) of orders l_j , l'_j ($l_j + m_j = l'_j + m'_j = 2m - 1$), completing the systems (2) and (4), respectively, to Dirichlet systems. The operator \mathfrak{A}^+ :

$$\mathfrak{D}(\mathfrak{A}^+) = W_2^{2m}(G), \quad \mathfrak{A}^+v = (\mathcal{L}^+v, B'_1v, \dots, B'_{mv}) \in K'_0 = L_2(G) \oplus \sum_{j=1}^m W_2^{2m-m'_j-1/2}(\Gamma)$$

has the same properties as the operator \mathfrak{A} . We denote its kernel by N^+ . From the fact that the codimension of $\mathfrak{R}(\mathfrak{A})$ is finite and from formula (3) one can obtain (see ⁽⁶⁾)

Lemma 1. Suppose

$$a_\mu(x) \in C^{2m+|\mu|}(\bar{G}), \quad b_{k\mu}(x) \in C^{\max(2m-1, 2m-m_k)}(\Gamma),$$

$$b'_{k\mu}(x) \in C^{\max(2m-1, 2m-m'_k)}(\Gamma),$$

and Γ is a surface of class C^{4m} . In order that the problem

$$\mathfrak{A}u = F \in K_0, \quad (5)$$

where $F = (f, \varphi_1, \dots, \varphi_m)$, have a solution, it is necessary and sufficient that for all $v \in N^+$ the equality

$$(f, v)_0 + \sum_{j=1}^m \langle \varphi_j, C'_{jv} \rangle_0 = 0 \quad (v \in N^+) \quad (6)$$

hold.

(We shall briefly write equality (6) as follows: $[F, N^+] = 0$.)

Denote by $\widetilde{W}_2^s(G)$ (s an arbitrary integer) the completion of the set of sufficiently smooth functions in the norm

$$\|u\|_s = \|u\|_s + \sum_{k=1}^{2m} \left\| \frac{\partial^{k-1}}{\partial \nu^{k-1}} u, \Gamma \right\|_{s-k+1/2} \quad (\nu \text{ is the normal to } \Gamma). \quad (7)$$

It is easy to see that if $s \geq 2m$, then the norms $\| \cdot \|_s$ and $\| \cdot \|_s$ are equivalent; if $s < 2m$, there is no such equivalence.

From the finite-dimensionality of N it follows that each element $u \in \widetilde{W}_2^s(G)$ can uniquely be represented in the form

$$u = u' + u''; \quad u' \in N; \quad (u'', N)_0 = 0. \quad (8)$$

The set of elements $\{u''\}$ forms a subspace $\widetilde{W}_2^s(G)$, which we shall denote by \widetilde{H}_s . Finally, let K_s^+ be the subspace of K_s consisting of elements $F \in K_s$ for which $[F, N^+] = 0$. We can now formulate the main theorem.

Theorem 1. *Consider the mapping $\Lambda : u \rightarrow (\mathcal{L}u, B_1u, \dots, B_mu)$ as an operator acting from \widetilde{H}_s into K_{s-2m}^+ . Then, under the corresponding smoothness assumptions, the closure $\overline{\Lambda}$ of the operator Λ establishes a homeomorphism between these spaces (for $s \geq 2m$, the homeomorphism is established by the operator Λ).*

The smoothness assumptions are the following:

- 1) for $s \geq 2m$, Γ is a surface of class

$$C^{2m+s}, \quad a_\mu(x) \in C^{2m+\max(|\mu|, s-2m)}(\overline{G}), \quad b_{k\mu}(x) \in C^{\max(s-1, s-m_k)}(\Gamma), \quad b'_{k\mu}(x) \in C^{\max(s-1, s-m'_k)}(\Gamma);$$

- 2) for $0 < s < 2m$, Γ is a surface of class C^{4m} ;

$$a_\mu(x) \in C^{2m+|\mu|}(\overline{G}); \quad b_{k\mu}(x) \in C^{\max(2m-1, 2m-m_k)}(\Gamma); \quad b'_{k\mu}(x) \in C^{\max(2m-1, 2m-m'_k)}(\Gamma);$$

$$c_{j\mu}(x) \in C^{\max(m_j + \frac{1}{2} - s, 0)}(\Gamma); \quad c'_{j\Gamma}(x) \in C^{\max(m_j + \frac{1}{2} - s, 0)}(\Gamma);$$

- 3) for $s \leq 0$, Γ is a surface of class C^{4m-s} ;

$$a_\mu(x) \in C^{2m+\max(|\mu|, -s)}(\overline{G});$$

$$b_{k\mu}(x) \in C^{\max(2m-1-s, 2m-m_k-s)}(\Gamma); \quad b'_{k\mu}(x) \in C^{\max(2m-1-s, 2m-m'_k-s)}(\Gamma);$$

$$c_{j\mu}(x) \in C^{m_j+1-s}(\Gamma); \quad c'_{j\mu}(x) \in C^{m_j+1-s}(\Gamma)$$

($c_{j\mu}(x)$, $c'_{j\mu}(x)$) are the coefficients of the expressions C_j , C'_j).

Remark. The expediency of introducing the norm (7) is explained by the following considerations. Let M be an arbitrary expression of order $k \leq 2m$ with sufficiently smooth coefficients; then it is easy to see that $\|Mu\|_{s-k} \leq C\|u\|_s$ ($u \in C^k(\overline{G})$), i.e. the mapping $u \rightarrow Mu$, after closure, acts continuously from all of $\widetilde{W}_2^s(G)$ into $W_2^{s-k}(G)$. Further, if B is an arbitrary boundary differential expression of order $k \leq 2m-1$, then $\|Bu, \Gamma\|_{s-k-1/2} \leq C\|u\|_s$ ($u \in C^k(\overline{G})$); consequently, the operator $u \rightarrow Bu|_\Gamma$ ($u \in C^k(\overline{G})$), after closure, acts continuously from all of $\widetilde{W}_2^s(G)$ into $W_2^{s-k-1/2}(\Gamma)$. In this sense we shall regard the expressions M and B as defined on all of $\widetilde{W}_2^s(G)$; if, for example, u_0 is a nonsmooth element of $\widetilde{W}_2^s(G)$, then by Mu_0 (Bu_0) we mean the value of the closure of the operator $u \rightarrow Mu$ ($u \rightarrow Bu|_\Gamma$) at the element u_0 .

3°. A function $u \in \widetilde{W}_2^k(G)$ (k an arbitrary integer) for which

$$\mathcal{L}u = f \in W_2^{k-2m}(G); \quad B_j u|_\Gamma = \varphi_j \in W_2^{k-m_j-1/2}(\Gamma) \quad (j = 1, \dots, m) \quad (9)$$

will be called a strong generalized solution of problem (9). (The application of the differential expressions \mathcal{L} and B_j to $u \in \widetilde{W}_2^k(G)$ is understood in the sense of the remark of item 2.)

Let G_1 be a subdomain of G adjacent to a piece Γ_0 of the surface Γ , let Γ_1 be the boundary of G_1 , and let $\overline{\Gamma_1} \cap \Gamma = \Gamma_0$. Let $\chi(x) \in C^\infty(\overline{G})$ vanish in some neighborhood in \overline{G} of the set $\overline{G} \setminus G_1$, and let near Γ_0 , $\partial\chi/\partial\nu = 0$. The following theorem is valid.

Theorem 2. Let $u \in \widetilde{W}_2^k(G)$ (k an arbitrary integer) be a strong generalized solution of problem (9). Suppose that in $G_1 \cup \Gamma_1$ the smoothness assumptions of Theorem 1 with $s = k+1$ are satisfied. If $\chi f \in W_2^{k-2m+1}(G_1)$, $\chi\varphi_j \in W_2^{k-m_j+1/2}(\Gamma_1)$, then $\chi u \in \widetilde{W}_2^{k+1}(G_1)$.

Theorem 2 is a theorem on the local increase of smoothness up to the boundary of strong generalized solutions.

4°. A function $u \in \widetilde{W}_2^s(G)$ (s an arbitrary integer) will be called a weak generalized solution of problem (9) if, for all $v \in W_2^{\max(2m, 2m-s)}(G)$, the equality

$$(u, \mathcal{L}^+ v)_0 + \sum_{j=1}^m \langle C_j u, B'_j v \rangle_0 = (f, v)_0 + \sum_{j=1}^m \langle \varphi_j, C'_j v \rangle_0 \quad (10)$$

is satisfied (the coefficients of the expressions \mathcal{L}^+ , B'_j , C'_j are assumed to be sufficiently smooth).

Lemma 2. *Suppose that the conditions of Theorem 1 are fulfilled. Then a weak generalized solution of problem (9) will be a strong one.*

From Theorem 2 and Lemma 2 there follows the assertion on the local increase of smoothness up to the boundary of weak generalized solutions of problem (9).

5°. If in (10) we put $\varphi_j = 0$ ($j = 1, \dots, m$), we obtain the definition of a weak generalized solution of the boundary-value problem with homogeneous boundary conditions. One may consider another definition of a generalized solution of such a problem (1).

Denote by $W_2^{2m}(\text{gr})^+$ the subspace of functions $v \in W_2^{2m}(G)$ for which $B_j'v|_\Gamma = 0$ ($j = 1, \dots, m$); $W_2^{2m+k}(\text{gr})^+ = W_2^{2m}(\text{gr})^+ \cap W_2^{2m+k}(G)$ ($k \geq 0$ an integer). $W_2^l(\text{gr})^+$ ($0 \leq l \leq 2m$) is the closure of the set $W_2^{2m}(\text{gr})^+$ in the metric $W_2^l(G)$. Finally, denote by $W_2^{-k}(\text{gr})^+$ ($k > 0$ an integer) the space with negative norm constructed from the null $L_2(G)$ and positive $W_2^k(\text{gr})^+$.

A function $u \in W_2^s(G)$ (s an arbitrary integer) is called a generalized solution of the problem

$$\mathcal{L}u = f \in W_2^{\min(s-2m, 0)}(\text{gr})^+ \quad (u \in (\text{gr})), \quad (11)$$

if, for all $v \in W_2^{\max(2m, 2m-s)}(\text{gr})^+$, the equality $(u, \mathcal{L}^+v)_0 = (f, v)_0$ is satisfied (we assume the coefficients of \mathcal{L}^+ to be sufficiently smooth).

Lemma 3. *Let $u \in W_2^s(G)$ (s an arbitrary integer) be a generalized solution of problem (11). If $f \in W_2^{s-2m}(G)$, then $u \in W_2^s(G)$ and satisfies relation (10) with $\varphi_j = 0$ ($j = 1, \dots, m$), i.e. is a generalized weak (and hence a strong) solution of problem (9).*

Theorem 3. *If $f \in W_2^{s-2m}(G)$, then for a generalized solution of problem (11) the assertion on the local increase of smoothness of the solution up to the boundary is valid: if $\chi f \in W_2^{s-2m+1}(G_1)$ and the smoothness conditions of Theorem 2 are fulfilled, then $\chi u \in \widetilde{W}_2^{s-2m+1}(G_1)$.*

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