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Abstract

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ON A BOUNDARY-VALUE PROBLEM FOR EQUATIONS OF MIXED TYPE

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In the present article a theorem is proved on the existence of a solution of the Hilbert–Poincaré problem for the Tricomi equation.

§ 1. Consider a domain Δ_+ in the half-plane $y > 0$, bounded by a curve γ having differentiable curvature, resting on the points $A(0,0)$ and $B(1,0)$, and by the segment of the straight line $y = 0$ enclosed between them. In the half-plane $y < 0$ consider a domain Δ_-, bounded by the same segment of the axis $y = 0$ and by two characteristics issuing from the points A and B. Denote by Δ the sum of the domains Δ_+ and Δ_- ($\Delta = \Delta_+ + \Delta_-$).

Consider the boundary-value problem P for the equation

$$yu_{xx} + u_{yy} = 0. \tag{E}$$

Equation (E) in the half-plane $y > 0$ is reduced to the canonical form

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta}u_\eta = 0 \tag{E_+}$$

by means of the substitution $\xi = x,\ \eta = \frac{2}{3}y^{3/2}$. In the half-plane $y < 0$, equation (E) is reduced to the form

$$u_{\xi\eta} - \frac{1}{6(\xi-\eta)}(u_\xi - u_\eta) = 0 \tag{E_-}$$

by the substitution $\xi = x - \frac{2}{3}(-y)^{3/2}$ and $\eta = x + \frac{2}{3}(-y)^{3/2}$.

The boundary conditions of problem P are as follows:

$$\text{On the curve } \gamma\ (y > 0) \qquad a\frac{\partial u}{\partial \vec{t}} + b\frac{\partial u}{\partial \vec{n}} + cu = f(t). \tag{1,1}$$

$$\text{On the characteristic } \xi = 0 \qquad u = \psi(\eta). \tag{1,2}$$

The derivatives are taken respectively with respect to arc length and to the normal to the curve γ, considered in the plane (ξ, η), where $0 \le t \le L$. In the course of the investigation we assume that the curve γ ends near the points $A(t = L)$ and $B(t = 0)$ in arbitrarily small arcs of the "normal curve" $(\xi - \frac{1}{2})^2 + \eta^2 = \frac{1}{4}$. This assumption, without restricting generality, considerably simplifies the investigation.

We shall seek regular solutions. By such a solution we shall mean a function: 1) continuous in the closed domain $\overline{\Delta}$; 2) having in Δ_+ second derivatives and satisfying equation (E); 3) having in Δ_- continuous partial derivatives with respect to x and y (or with respect to ξ and η) and representable by means of Darboux' s formula

$$u = \frac{\Gamma(1/3)}{\Gamma^2(1/6)} \int_\xi^\eta \tau(t) \frac{(\eta - \xi)^{2/3} dt}{(\eta - t)^{5/6} (t - \xi)^{5/6}} - \frac{(4/3)^{2/3} \Gamma(2/3)}{2\Gamma^2(5/6)} \int_\xi^\eta \nu(t) (\eta - t)^{-1/6} (t - \xi)^{-1/6} dt; \tag{1,3}$$

4) satisfying on AB $(y = 0)$ the gluing conditions:

$$\lim_{y \to +0} u(x, y) = \lim_{y \to -0} u(x, y) = \tau(x), \qquad \lim_{y \to +0} u_y = \lim_{y \to -0} u_y = \nu(x).$$

It follows from item 3) that u in Δ_- satisfies equation (E), if it has second derivatives. If it does not have them, then u is a generalized solution of (E).

Let us note that the question of the existence of a generalized solution in a broader sense is more trivial.

The problem under consideration, after using the results of K. I. Babenko ([1]), reduces to a problem with zero data on the characteristic and with condition (1,1) on γ, in which the right-hand side is the sum of the initial function and terms connected with a certain auxiliary solution of the Tricomi problem. In what follows we shall consider precisely this problem, retaining the notation of condition (1,1).

The method of investigation proposed below is general and is suitable for any equation of mixed type. It consists in the following: suppose that we have solved the Tricomi problem, i.e. have found in Δ_+ a solution of equation (E) satisfying the conditions: $U|_{z=0} = 0$ and $U|_\gamma = \varphi(s)$. Then, by means of the solution u, we shall try to satisfy the conditions of problem P. For this it is necessary to choose $\varphi(s)$ in such a way that condition (1,1) be fulfilled; we obtain a singular equation for φ', and problem P is reduced to the study of this equation. In the course of the investigation considerable difficulties arise, connected with obtaining estimates for the kernel of the singular equation.

This method makes it possible to obtain information about the solvability conditions of problem P, to establish the class of solutions, etc.

§ 2. Let us pass to the investigation of problem P. In solving the Tricomi problem in Δ_+, the desired solution is usually represented in the form

$$u(x,y) = -\int_0^1 \nu(x_1) G(x_1, 0, x, y)\, dx_1 - \int_\gamma \varphi(s) \frac{\partial G}{\partial \nu}\, ds, \tag{2,1}$$

where the derivative with respect to the conormal is

$$\partial/\partial\nu = y\cos(nx)\,\partial/\partial x + \cos(ny)\,\partial/\partial y,$$

and G is the Green' s function of the following problem: to find a solution of equation (E) in Δ_+ satisfying the conditions

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \nu(x) \quad (0 < x < 1)$$

and

$$u|_\gamma = \varphi(s).$$

§ 3. Denote the second term of formula (2,1) by u_1

$$u_1 = -\int_0^1 \varphi(s) \frac{\partial G}{\partial \nu}\, ds \tag{3,1}$$

and find $\partial u_1/\partial t$ and $\partial u_1/\partial n$. Calculation of the normal derivative of $u_1(\xi,\eta)$ shows that on the contour γ it can be represented in the form

$$\left(\frac{\partial u_1}{\partial n}\right)_- = \int_0^L \left[\frac{1}{\pi(t-t_1)} + H(t,t_1)\right] \varphi'(t)\, dt, \tag{3,2}$$

where

$$H(t,t_1) = O\left(\frac{1}{\eta_0(t_1)}\right) + O\left(\ln^2\left|\frac{t-t_1}{t+t_1}\right|\right). \tag{3,3}$$

The principal difficulty in this investigation consists in obtaining the following estimate. Since $u_1|_\gamma = \varphi(t_1)$, it follows that $(\partial u_1/\partial t_1) = \varphi'(t_1)$. Denote

$$w(\xi,\eta) = -\int_0^1 G(\xi,\eta,\xi_0,0)\,\nu(\xi_0)\, d\xi_0 \tag{3,4}$$

and investigate the derivatives of this function. w is an operator from ν, which in turn can be expressed through φ. Therefore, first, from the gluing condition –

we shall find a relation between φ and ν and only after this shall we return to finding and estimating the derivatives of the function w.

§ 4. In Δ_- the solution of problem P is given by the Riemann–Hadamard formula (equivalent to Darboux' s formula). In our case, when $\psi(\eta) = 0$, this formula (for $y = 0$) gives

$$u(x, 0) = k \int_0^x \nu(u)(x - u)^{-1/3}\, du.$$

On the other hand, in Δ_+ the solution is given by formula $(2, 1)$ and on the line $y = 0$ takes the form

$$u(x, 0) = -\int_0^1 \nu(u)G(x, 0, u, 0)\, du - \int_0^L \varphi(u)\lambda(u, x, 0)\, du.$$

Applying the gluing condition on the line of degeneration $(y = 0)$, substituting the value of $G(x, 0, u, 0)$, and applying, to the equation obtained, the Tricomi operator

$$\frac{d}{dy} \int_0^y (y - x)^{-2/3} \ldots\, dx,$$

we obtain a singular integral equation, which is then regularized.

The equation obtained in this way, as was shown by Tricomi ([2]), is solvable for any right-hand side, and this solution is represented in the form

$$\nu(x) = \int_0^L \varphi'(t)\Phi(x, t)\, dt. \tag{4,1}$$

Here

$$|\Phi(x, u)| \leq c(1 - x)^{-1/3}\left[1 + x^\delta (L - u)^{-2/3-\delta}\right], \qquad \text{where } 0 < \delta < 1/3. \tag{4,2}$$

§ 5. Let us now return to formula $(3, 4)$. Since on the contour $G(x, y, x_1, 0) = 0$, the derivative $(\partial w/\partial t_1)_- = 0$. Therefore it is necessary to find only $\partial w/\partial n$ on the contour (which we shall denote by $(\partial w/\partial n)_-$).

$$\left(\frac{\partial w}{\partial n}\right)_- = -\int_0^1 \nu(x_1)\frac{\partial}{\partial n}G(x,y,x_1,0)\,dx_1 = \int_0^1 \varphi'(t)\Phi_1(t,t_1)\,dt, \qquad (5{,}1)$$

where the estimate

$$|\Phi_1(t,t_1)| < c\,[\eta^{-1/3}(t_1) + (L-t)^{-2/3-\delta}\eta(t_1)^{\delta-1/3}]$$

holds for small $L-t_1$, and an analogous one for small t_1. Using the results of §§ 3 and 5, one can write the boundary condition on γ in the following form:

$$a(t_1)\varphi'(t_1) + \frac{b(t_1)}{\pi}\int_0^L \frac{\varphi'(t)}{t-t_1}\,dt + \int_0^L K(t,t_1)\varphi'(t)\,dt = f(t_1). \qquad (5{,}2)$$

Suppose that the coefficient $b(t_1)$ vanishes at the ends of the interval of integration, more precisely:

$$|b(t_1)| < ct_1^{\alpha}, \qquad (5{,}3)$$

where $0 < \alpha \leq 1$ near the point $B(1,0)$, and

$$|b(t_1)| < C(L-t_1)^{\alpha} \qquad (5{,}4)$$

near the point $A(0,0)$. By virtue of these assumptions and the estimates obtained for the various parts of the kernel $K(t,t_1)$, for small t_1 we shall have

$$|K(t,t_1)| < \frac{c}{t_1^{1-\alpha}} + ct_1^{\alpha}\ln^2\left|\frac{t-t_1}{t+t_1}\right| + c(L-t)^{-2/3-\delta}t_1^{\alpha+\delta-1/3}. \qquad (5{,}5)$$

For small $L-t_1$ it is necessary in $(5,5)$ to replace t_1 by $L-t_1$.

For $\delta > 0$, $K(t,t_1)$ will be a kernel with a weak singularity.

§ 6. Let us consider the question of regularizing equation (5.2). Applying the well-known theory of singular integral equations, we arrive at the conclusion that equation (5.2) can be regularized and reduced to the equation

$$\varphi'(t_1) + \int_0^L h_2(t,t_1)\varphi'(t)\,dt = f_1(t_1), \qquad (6.1)$$

or, putting $(L-t)^{-1/3}t^{-1/3}\varphi'(t) = h(t)$, to the equation

$$h(t_1) + \int_0^L H_1(t,t_1)h(t)\,dt = f_2(t_1). \qquad (6.2)$$

Here the kernel H_1 will satisfy the inequality

$$\int_0^L \int_0^L |H_1(t, t_1)|^2 \, dt \, dt_1 < \infty,$$

which follows directly from estimate (5.5) of the kernel $K(t, t_1)$. The right-hand side of (6.2) will be square-integrable. Thus, Fredholm theory is applicable to (6.2).

If the index χ of equation (5.2) satisfies the condition $\chi \geq 0$, then no additional restrictions are required for obtaining equation (6.1) (see ([3])), and the question reduces to the solvability of equation (6.1).

Assuming that $\chi = 0$ and that the solution is unique, we obtain that equation (6.1) has a solution if $f(t)$ is a continuous function.

The function $\varphi(t)$ is determined up to an additive constant, but since $\varphi(L) = 0$, the constant is determined uniquely. Thus the theorem has been proved.

Theorem. *If for $0 < t < L$ the function $b(t)$ satisfies conditions (5.3), (5.4), $a(t)$ and $b(t)$ satisfy Hölder conditions, $\chi = 0$, and the homogeneous problem has a unique solution, then there exists a solution of problem P for every continuous right-hand side $f(t)$.*

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CITED LITERATURE

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3. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.