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1964

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Abstract

Full Text

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PARTIAL TOPOLOGICAL PRODUCTS

(Presented by Academician P. S. Aleksandrov on 20 IX 1963)

In the present note partial topological products are introduced, generalizing the ordinary (Tychonoff) topological products (which I shall call complete). Partial products possess many interesting properties and turn out to be well applicable to the construction of universal (in the topological sense) spaces of a given weight and a given dimension, to open-closed zero-dimensional mappings that raise dimension, to finite-to-one closed mappings; to inverse (polyhedral) spectra, to the extension of zero-dimensional mappings to bicompact extensions, etc. All the results of the notes ([1,2]) were obtained with the aid of partial products. In note ([2]) partial products are called local products (i.e. in that note the words "local product" must everywhere be replaced by the words "partial product").

Definition 1. Let a topological space X_0 be given, with an open set ${}_\alpha O_0$ distinguished in it, and a space Z_α. By the **partial product** $X_\alpha = P(X_0, Z_\alpha, {}_\alpha O_0)$ of the **base** X_0 by the **fiber** Z_α with respect to the **open set** ${}_\alpha O_0$ we shall mean a topological space consisting of two disjoint sets ${}_\alpha O_\alpha$ and $X_\alpha \setminus {}_\alpha O_\alpha$, where the set $X_\alpha \setminus {}_\alpha O_\alpha$ is homeomorphic to the set $X_0 \setminus {}_\alpha O_0$ (this homeomorphism will be denoted by ${}^\alpha_0\mathfrak{F}$), and the set ${}_\alpha O_\alpha$ is homeomorphic to the complete topological product ${}_\alpha O_0 \times Z_\alpha$*; moreover, the natural projection of the complete product

$${}_\alpha O_\alpha \equiv {}_\alpha O_0 \times Z_\alpha$$

onto the factor ${}_\alpha O_0$ will also be denoted by ${}^\alpha_0\mathfrak{F}$. As basic open sets in X_α we shall take: 1) inverse images of open subsets in X_0 under the mapping ${}^\alpha_0\mathfrak{F} : X_\alpha \to X_0$; 2) open subsets of the set ${}_\alpha O_\alpha$, regarded as the complete topological product ${}_\alpha O_0 \times Z_\alpha$.

Definition 2. Let a space X_0 be given, a system of its open subsets ${}_\alpha O_0$, $\alpha \in \mathfrak{A}$, and a system of spaces Z_α, $\alpha \in \mathfrak{A}$. Then the partial products

$$X_\alpha = P(X_0, Z_\alpha, {}_\alpha O_0)$$

and the mappings

$${}^0_\alpha\mathfrak{F} : X_\alpha \to X_0, \qquad \alpha \in \mathfrak{A},$$

are defined. By the **partial topological product**

$$X_\mathfrak{A} = P(X_0, \{Z_\alpha\}, \{{}_\alpha O_0\}, \alpha \in \mathfrak{A})$$

of the **base** X_0 by the **system of fibers** Z_α with respect to the **system of open sets** ${}_\alpha O_0$, $\alpha \in \mathfrak{A}$, we shall mean the space whose points are all possible collections

$$x_{\mathfrak{A}} = \{x_\alpha\}$$

of points $x_\alpha \in X_\alpha$ (one point from each X_α) satisfying, for any two indices α' and $\alpha'' \in \mathfrak{A}$, the relation

$$ {}^{0}_{\alpha'}\mathfrak{F}(x_{\alpha'}) = {}^{0}_{\alpha''}\mathfrak{F}(x_{\alpha''}).$$

The mapping which assigns to each point $x_{\mathfrak{A}} = \{x_\alpha\} \in X_{\mathfrak{A}}$ the point $x_\alpha \in X_\alpha$ (the α-th coordinate of the point $x_{\mathfrak{A}}$) will be denoted by

$$ {}^{\mathfrak{A}}_{\alpha}\mathfrak{F},$$

i.e.

$$x_\alpha = {}^{\mathfrak{A}}_{\alpha}\mathfrak{F}(x_{\mathfrak{A}}).$$

We define the topology in $X_{\mathfrak{A}}$ so that the collection of all possible sets

$$\left({}^{\mathfrak{A}}_{\alpha}\mathfrak{F}\right)^{-1}(V_\alpha), \qquad \alpha \in \mathfrak{A},$$

where V_α is an open set in X_α, forms a subbase.

Remark 1. If ${}_{\alpha}O_0 \equiv X_0$ for all $\alpha \in \mathfrak{A}$, then

$$P(X_0, \{Z_\alpha\}, \{{}_{\alpha}O_0\}, \alpha \in \mathfrak{A}) \equiv X_0 \times \prod_\alpha Z_\alpha,$$

i.e. partial products generalize complete ones.

* The set ${}_{\alpha}O_\alpha$ will simply be identified with the product ${}_{\alpha}O_0 \times Z_\alpha$.

Properties of Partial Products

1. The mappings ${}^{\alpha}_{0}\mathfrak{F}$ and ${}^{\mathfrak{A}}_{\alpha}\mathfrak{F}$, $\alpha \in \mathfrak{A}$, are continuous, i.e. the mapping $\mathfrak{A}\mathfrak{F} = {}^{\alpha'}_{0}\mathfrak{F} \cdot {}^{\alpha''}_{\alpha'}\mathfrak{F} \cdot {}^{\mathfrak{A}}_{\alpha''}\mathfrak{F} : X_{\mathfrak{A}} \to X_0$ is continuous, where α' and α'' are arbitrary indices from $\mathfrak{A}$.
2. For every point $x_{\mathfrak{A}} \in X_{\mathfrak{A}}$ there exists a mapping $f_{x_{\mathfrak{A}}} : X_0 \to X_{\mathfrak{A}}$ such that $x_{\mathfrak{A}} \in f_{x_{\mathfrak{A}}}(X_0)$, the set $f_{x_{\mathfrak{A}}}(X_0)$ is closed in $X_{\mathfrak{A}}$, and the mapping ${}^{\mathfrak{A}}_{0}\mathfrak{F} \cdot f_{x_{\mathfrak{A}}}$ coincides with the identity mapping of the space X_0. In other words, through every point $x_{\mathfrak{A}} \in X_{\mathfrak{A}}$ there passes a secant surface. From this fact follow the relations $\dim X_{\mathfrak{A}} \geq \dim X_0$, $\operatorname{ind} X_{\mathfrak{A}} \geq \operatorname{ind} X_0$, $\operatorname{Ind} X_{\mathfrak{A}} \geq \operatorname{Ind} X_0$.
3. If the base X_0 and all fibers Z_α are T_i-spaces, $i = 0, 1, 2$, then $X_{\mathfrak{A}}$ is respectively the same. If X_0 and all Z_α, $\alpha \in \mathfrak{A}$, are (completely) regular, then $X_{\mathfrak{A}}$ is respectively the same.
4. If X_0 and all Z_α, $\alpha \in \mathfrak{A}$, are bicompact (and Hausdorff), then the partial product $X_{\mathfrak{A}}$ is the same.
5. $w(X_{\mathfrak{A}}) \leq \max(w(X_0), \max_\alpha(w(Z_\alpha)), m(\mathfrak{A}))^*$.

6. If $\operatorname{ind} Z_\alpha = 0$ for every $\alpha \in \mathfrak{A}$, then $\operatorname{ind} X_\mathfrak{A} = \operatorname{ind} X_0$.
7. Suppose that the fiber Z_α of the partial product $X_\alpha = P(X_0, Z_\alpha, {}_\alpha O_0)$ consists of τ isolated points. If the base X_0 is (perfectly) normal, paracompact, metrizable, then X_α respectively is the same. Moreover, if the set ${}_\alpha O_0$ is of type F_σ, then $\dim X_\alpha = \dim X_0$, and if X_0 is perfectly normal, then $\operatorname{Ind} X_\alpha = \operatorname{Ind} X_0$.
8. If all fibers Z_α, $\alpha \in \mathfrak{A}$, consist of isolated points, and the system $\{{}_\alpha O_0\}$, $\alpha \in \mathfrak{A}$, is locally finite, then the partial product $X_\mathfrak{A}$:
   a) is locally normal, if the base X_0 is normal; if, in addition, each set ${}_\alpha O_0$ is of type F_σ, then $\operatorname{loc}\dim X_\mathfrak{A} \leq \dim X_0$, i.e. $\dim X_\mathfrak{A} = \dim X_0$, if $X_\mathfrak{A}$ is weakly paracompact;
   b) is paracompact with $\dim X_\mathfrak{A} = \dim X_0$, if X_0 is paracompact;
   c) is metrizable with $\dim X_\mathfrak{A} = \dim X_0$, if the base X_0 is metrizable (in this item the system $\{{}_\alpha O_0\}$, $\alpha \in \mathfrak{A}$, may even be assumed locally countable).
9. If all fibers Z_α of the partial product $X_\mathfrak{A} = P(X_0, \{Z_\alpha\}, \{{}_\alpha O_0\}, \alpha \in \mathfrak{A})$ are bicompact, then:
   a) all mappings ${}^{\mathfrak{A}}_{0}\mathfrak{F}$ and ${}^{\mathfrak{A}}_{\alpha}\mathfrak{F}$, $\alpha \in \mathfrak{A}$, are closed and bicompact, and $X_\mathfrak{A}$ coincides with the space of such a partition $\omega_\mathfrak{A}$ of the complete product $X_0 \times \prod_\alpha Z_\alpha$, that the points $(x_0', \{z_\alpha'\})$ and $(x_0'', \{z_\alpha''\})$ are contained in one element of the partition $\omega_\mathfrak{A}$ if and only if $x_0' = x_0''$ and $z_\alpha' = z_\alpha''$ for $x_0' = x_0'' \in {}_\alpha O_0$;
   b) $X_\mathfrak{A}$ is (locally) bicompact, finally compact, (weakly, strongly) paracompact, if respectively the base X_0 is so. Moreover, if the base X_0 is paracompact and all fibers Z_α are bicompact, then $\dim X_\mathfrak{A} \leq \dim X_0 + \sum_\alpha \dim Z_\alpha$. In particular, if the fibers Z_α are zero-dimensional bicompacts, then $\dim X_\mathfrak{A} = \dim X_0$.
10. The partial product $X_\mathfrak{A} = P(X_0, \{Z_\alpha\}, \{{}_\alpha O_0\}, \alpha \in \mathfrak{A})$ is the limit of the spectrum $S_{\aleph_0} = \{X_{(\alpha_1 \ldots \alpha_s)}, {}^{(\alpha_1 \ldots \alpha_s)}_{(\alpha_1 \ldots \alpha_k)}\mathfrak{F}\}, \{\alpha \in \mathfrak{A}\}$, of "elementary" partial products $X_{(\alpha_1 \ldots \alpha_s)} = P(X_0, \{Z_{\alpha_i}\}, \{{}_{\alpha_i} O_0\}, i = 1, \ldots, s)$, where ${}^{(\alpha_1 \ldots \alpha_k)}_{(\alpha_1 \ldots \alpha_s)}\mathfrak{F}$ denotes the naturally arising mapping of the partial product $X_{(\alpha_1, \ldots, \alpha_s)}$ onto the partial product $X_{(\alpha_1 \ldots \alpha_k)}$ for $\{\alpha_1, \ldots, \alpha_k\} \subseteq \{\alpha_1, \ldots, \alpha_s\}$**.

* $w(X)$ denotes the weight of the space X, $m(X)$ denotes the cardinality of the set X.

** In the article (2), property 10 was used as the definition of partial products.

Theorem 1. The partial product $X_\mathfrak{A} = P(I^n, \{Z_\alpha\}, \{{}_\alpha O_0\}, \alpha \in \mathfrak{A})$, where I^n denotes the n-dimensional cube and the system $\{{}_\alpha O_0\}$, $\alpha \in \mathfrak{A}$, is an arbitrary countable base in I^n, will be a universal space:

a) for all spaces of weight τ, and only for them, that are at most n-dimensional in the sense of dim metric spaces, if each layer Z_α, $\alpha \in \mathfrak{A}$, consists of τ isolated points;

b) for all completely regular spaces possessing separating mappings of weight $\leqslant \tau$ ([1]) onto n-dimensional metric spaces with a countable base, and only for them, if each layer Z_α is homeomorphic to D^τ, i.e. is the full product of τ spaces consisting of two isolated points. In particular, $X_\mathfrak{A}$ will be a universal space for all n-dimensional in the sense of dim metric spaces of weight τ and for all n-dimensional in the sense of dim bicompacts of weight τ, zero-dimensionally mapped onto compacta (and also for all the spaces listed in item 2) of Theorem 1 from ([1]).

The partial product $X_\mathfrak{A}$ from items a) and b) satisfies the conditions

$$w(X_\mathfrak{A}) = \tau \quad \text{and} \quad \dim X_\mathfrak{A} = \operatorname{ind} X_\mathfrak{A} = \operatorname{Ind} X_\mathfrak{A} = n.$$

Theorem 2. The partial product $X_\mathfrak{A} = P(I^\infty, \{Z_\alpha\}, \{{}_\alpha O_0\}, \alpha \in \mathfrak{A})$, where I^∞ is the Hilbert brick, and the system $\{{}_\alpha O_0\}$ forms a countable base in I^∞, will be a universal space:

a) for all metric spaces of weight τ, and only for them, if each layer Z_α, $\alpha \in \mathfrak{A}$, consists of τ isolated points;

b) for all completely regular spaces possessing a separating mapping of weight τ onto a metric space with a countable base, and only for them, if each layer Z_α is homeomorphic to D^τ. In particular, $X_\mathfrak{A}$ will be a universal space for all metric spaces and all bicompacts, zero-dimensionally mapped onto compacta, of weight τ.

For the product X_α from items a) and b) the relation

$$w(X_\mathfrak{A}) = \tau$$

will hold.

Remark 2. a) If in item a) of Theorem 1 $n = 0$, then $X_\mathfrak{A}$ coincides* with the full product of a countable number of spaces consisting of τ isolated points, i.e. $X_\mathfrak{A}$ in this case coincides with the generalized Baire space of weight τ.

b) For $\tau = \aleph_0$, item b) of Theorem 1 can be strengthened as follows:

The partial product $P(I^n, \{Z_\alpha\}, \{{}_\alpha O_0\}, \alpha \in \mathfrak{A})$, where the system $\{{}_\alpha O_0\}$ is a countable base in I^n, and each layer Z_α consists of two isolated points, will be a universal space for all n-dimensional metric spaces with a countable base**.

The existence of a universal space among n-dimensional metric spaces of weight τ was first shown in ([3]).

Theorem 3. The partial product $P^\tau_\chi = P(I^\chi, \{Z_\alpha\}, \{{}_\alpha O_0\}, \alpha \in \mathfrak{A})$, where I^χ is the Tikhonov brick of weight χ, the system $\{{}_\alpha O_0\}$, $\alpha \in \mathfrak{A}$, forms a base in I^χ, and each layer Z_α is homeomorphic to D^τ, $\tau \geqslant \chi$, will be a universal space for all completely regular spaces possessing a separating mapping of weight τ onto a completely regular space of weight χ, and only for them. In particular, P_χ^τ will be universal for all bicompacts of weight τ possessing a zero-dimensional mapping onto bicompacts of weight χ, and only for them. P_χ^τ is a bicompactum of weight τ, zero-dimensionally mapped onto I^χ.

The following theorem reveals the connections of partial products with locally trivial fiber spaces.

Theorem 4. Let a space X_0 and its covering $\nu = \{{}_\alpha O_0\}$, $\alpha \in \mathfrak{A}$, be given. The space X of any such locally trivial fiber space

* Since X_0 consists of one point.
** It is useful to compare this assertion with Theorem 1 from ([2]).

$p : X \to X_0$ with base X_0 and fiber Z, such that each set $p^{-1}(\alpha O_0)$, $\alpha \in \mathfrak{A}$, coincides with the full product $\alpha O_0 \times Z$ (and the mapping p on the set $p^{-1}(\alpha O_0)$ coincides with the projection of the full product $\alpha O_0 \times Z$ onto the factor αO_0), has a homeomorphic mapping

$$f_{\mathfrak{A}} : X \to P(X_0, \{Z_\alpha\}, \{{}_\alpha O_0\}, \alpha \in \mathfrak{A}),$$

where $p = {}_0^{\mathfrak{A}}\mathfrak{F} \cdot f_{\mathfrak{A}}$ and $Z_\alpha \equiv Z$, $\alpha \in \mathfrak{A}$.

The following results strengthen and refine Theorem 3 of ([1]) and Theorem 8 of ([2]). In particular, Theorem 5 definitively settles (in the sense of the dimensions of preimages and images) the question of open-closed zero-dimensional mappings of metric spaces.

Theorem 5. Every metric space R of weight τ and with $\dim R > 0$ is an open, closed, bicompact, and zero-dimensional image of a one-dimensional, in the sense of dim, metric space S_R of weight τ, where $\operatorname{ind} S_R = 0$, if $\operatorname{ind} R = 0$.

Theorem 6. a) Every completely regular T_1-space X possessing a decomposing mapping of weight τ onto a completely regular space of weight χ is an open, closed, bicompact, and zero-dimensional image under the mapping f_X of a completely regular space Y_X of weight τ, possessing a decomposing mapping of weight τ onto a completely regular space of weight χ, where Y_X is a subset of a one-dimensional, in the sense of dim, bicompactum of weight τ, and $w(f_X^{-1}(x)) \leq \chi$ for every point $x \in X$. If the space X is paracompact, then the same is true of the space Y_X, and then

$$\dim Y_X \leq \dim X.$$

b) If in item a) the space X is strongly paracompact, or finally compact, or bicompact, then, respectively, the space Y_X will also be the same, and then

$$\dim Y_X = 1.$$

c) In particular, if a bicompactum X of weight τ has a zero-dimensional mapping onto a bicompactum of weight χ, then X is an open and zero-dimensional image under the mapping f_X of a one-dimensional, in the sense of dim, bicompactum Y of weight τ, mapped zero-dimensionally onto a bicompactum of weight χ, where each set $f_X^{-1}(x)$, $x \in X$, has weight χ.

Note that for $\chi = \aleph_0$ all the sets $f_X^{-1}(x)$, $x \in X$, in items a)–c) will be metrizable, and in items b) and c) the space Y_X will satisfy the relations

$$\dim Y_X = \operatorname{ind} Y_X = \operatorname{Ind} Y_X = 1$$

(and in this case, in item b), the space X may be regarded as completely paracompact ([4])).

Theorem 7. The following propositions are equivalent:

a) A bicompactum X of weight τ has a zero-dimensional mapping onto a compactum and $\dim X = n$.

b) The bicompactum X is the limit of a spectrum

$$S = \{P_\alpha, {}^{\beta}_{\alpha}\mathfrak{F}\}, \quad \alpha \in \mathfrak{A}, \quad m(\mathfrak{A}) = \tau,$$

of n-dimensional polyhedra P_α, taken in certain triangulations K_α, where each projection ${}^{\beta}_{\alpha}\mathfrak{F}$ is non-degenerate (i.e. finite-to-one) and simplicial with respect to the triangulation K_β and to some (not necessarily one-fold) barycentric subdivision of the triangulation K_α. The set $\mathfrak{A}$ has a minimal index 0, by which the n-dimensional simplex E^n, taken in its natural triangulation K_0, is numbered. Each projection ${}^{\beta}_{\alpha}\mathfrak{F}$ is represented as a superposition of two-fold projections

$${}^{\alpha_1}_{\alpha}\mathfrak{F},\ {}^{\alpha_2}_{\alpha_1}\mathfrak{F}, \ldots,\ {}^{\beta}_{\alpha_s}\mathfrak{F}.$$

For $\tau = \aleph_0$, the set $\mathfrak{A}$ may be regarded as the natural sequence.

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Received
18 IX 1963

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Note: Figure translations are in progress. See original paper for figures.