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Abstract

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MATHEMATICS

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ON A MIXED PROBLEM FOR AN EQUATION NOT SOLVED WITH RESPECT TO THE HIGHEST TIME DERIVATIVE

(Presented by Academician S. L. Sobolev on V 4, 1964)

Consider the equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+2\frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

and the conditions

$$u\Big|_{t=0}=u_0(x,y),\qquad \frac{\partial u}{\partial t}\Big|_{t=0}=u_1(x,y), \tag{2}$$

$$u\Big|_{\Gamma}=0, \tag{3}$$

where Γ is a smooth contour bounding a closed, simply connected domain Ω in the x, y-plane.

If u_0, u_1 belong to C_2 and vanish on the boundary Γ, then the solution of problem (1), (2), (3) exists for all t, $(x,y)\in\overline{\Omega}$, is analytic in t, and is twice continuously differentiable with respect to x, y in Ω. This assertion is easily proved by determining u_i from (1), (2), where

$$u=\sum u_i(x,y)t^i, \tag{4}$$

with

$$u_i\Big|_{\Gamma}=0.$$

We shall call a number λ an eigenvalue of the equation

$$L_\lambda u = \lambda^2 \Delta u + 2\lambda \frac{\partial^2 u}{\partial x\, \partial y} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{5}$$

if there exists a function $u \in \overset{0}{W}{}_2^1(\Omega)$ such that

$$\int_\Omega u\, L_\lambda \psi \, d\Omega = 0 \tag{6}$$

for all infinitely differentiable functions ψ that are finite in Ω. The eigenvalues of equation (5), when a complete system of eigenfunctions exists, characterize the behavior as $t \to \infty$ of solutions of the mixed problem.

Mixed problems for equations of the form

$$\frac{\partial^2}{\partial t^2} Au + Mu = 0,$$

where A is an elliptic operator of the second order and M is a differential operator of the same order, were considered in works ($^{1-6}$). The study of the behavior as $t \to \infty$ of the solutions of these problems leads to the study of a certain bounded self-adjoint operator in a Hilbert space, whose spectrum coincides with those values of λ for which the operator $(\lambda A + M)u$ is hyperbolic.

For this operator, in some cases it is possible to construct a system of generalized eigenfunctions (^{2}), invariant subspaces in which it is easy to give a simple representation of the operator and to investigate

the asymptotics of the corresponding solutions of the mixed problem (^{5},^{6}). In the case of equation (5) the spectrum coincides with the complex values of λ for which this equation is elliptic, i.e., the characteristic polynomial has complex and distinct roots. The Dirichlet problem in this case turns out to be non-Fredholm. Equations of this type, in the case where Ω is an ellipse, were studied by N. E. Tovmasyan. He gave criteria for the existence of nontrivial solutions of the homogeneous Dirichlet problem and a representation for such solutions. To construct a complete system of eigenfunctions of equation (5), it is more convenient for us to construct systems of solutions of this equation that vanish on the boundary of the circle and are polynomials in x and y. By analogous reasoning one can easily obtain the eigenfunctions of the equation

$$\lambda^2 M_1 u + \lambda M_2 u + M_3 u = 0, \tag{5'}$$

if M_1, M_2, M_3 are linear differential operators of the second order with constant coefficients, containing only second derivatives with respect to x and y, and Ω is an ellipse arbitrarily situated in the plane (this case is easily reduced to the case of a circle), and moreover the type of equation $(5')$ plays no role.

The general solution of equation (5) for any λ has the form

$$u = \varphi(x + \mu_1 y) + \psi(x + \mu_2 y), \tag{7}$$

where μ_1, μ_2 are the roots of the equation

$$\lambda^2(1 + \mu^2) + 2\lambda\mu + \mu^2 = 0. \tag{8}$$

Thus,

$$u = \varphi\left(x - \frac{\lambda - i\lambda^2}{1 + \lambda^2}\, y\right) + \psi\left(x - \frac{\lambda + i\lambda^2}{1 + \lambda^2}\, y\right).$$

Putting $z = x + iy,\ \bar{z} = x - iy$, we have

$$u = \varphi_1\left(z + \frac{1 - \lambda i}{1 + \lambda i + 2\lambda^2}\, \bar{z}\right) + \psi_1\left(z + \frac{1 - \lambda i + 2\lambda^2}{1 + \lambda^2}\, \bar{z}\right). \tag{9}$$

Let

$$\lambda = \frac{1}{2i}\left(\frac{1}{\tau} - 1\right);$$

then

$$u = \varphi_1(z + \tau\bar{z}) + \psi_1\left(z + \left(2 - \frac{1}{\tau}\right)\bar{z}\right). \tag{10}$$

Let Ω be the circle $z\bar{z} = 1$. We shall construct functions of the form (10) that vanish for $z\bar{z} = 1$, i.e., eigenfunctions of equation (5). To this end we prove that there exists a polynomial P_n of degree $n - 2$, depending only on $z + \tau\bar{z}$, such that

$$P_n(z + \tau\bar{z})\big|_{z\bar{z}=1} = \{(z + \tau\bar{z})^n - z^n - \tau^n \bar{z}^n\}_{z\bar{z}=1}. \tag{11}$$

It is obvious that $P_1 \equiv 0,\ P_2 \equiv 2\tau$. Suppose that there exist polynomials $P_1,\ P_2, \dots, P_n$ possessing the indicated properties. Then

$$\begin{aligned}
&(z + \tau\bar{z})^{n+1} - z^{n+1} - \tau^{n+1}\bar{z}^{n+1}\big|_{z\bar{z}=1}\\
&= \{(z + \tau\bar{z})[(z + \tau\bar{z})^n - (z^n + \tau^n\bar{z}^n)] + \tau(z^{n-1} + \tau^{n-1}\bar{z}^{n-1})\}_{z\bar{z}=1}\\
&= \{(z + \tau\bar{z})P_n + \tau(z + \tau\bar{z})^{n-1} - \tau P_{n-1}\}_{z\bar{z}=1}.
\end{aligned}$$

Putting

$$P_{n+1} = (z + \tau\bar{z})P_n + \tau(z + \tau\bar{z})^{n-1} - \tau P_{n-1}, \tag{12}$$

we obtain recurrence formulas for determining P_n. Consider the polynomials

$$Q_n(z + \tau\bar{z}) = (z + \tau\bar{z})^n - P_n(z + \tau\bar{z}) \tag{13}$$

and the functions

$$u_n^k = Q_n(z + \tau\bar{z}) - Q_n(z + \tau e^{-2\pi ik/n}\bar{z}). \tag{14}$$

Obviously, $u_n^k\big|_{z\bar{z}=1} = 0$, and if $\tau = \tau_n^k$, where

$$\tau_n^k = \left(2 - \frac{1}{\tau_n^k}\right) e^{2\pi ik/n}, \tag{15}$$

then u_n^k also satisfy equation (5) for $\lambda = \lambda_n^k = \dfrac{1}{2i}\left(\dfrac{1}{\tau_n^k} - 1\right)$, i.e., u_n^k are eigenfunctions of equation (5) corresponding to λ_n^k. For each n, equation (15) has $2(n-1)$ distinct roots, and to them correspond $2(n-1)$ distinct eigenvalues for $k = 1, 2, \dots, n-1$. We shall prove that, by a linear combination with complex coefficients of the vectors $(u_n^k, \lambda_n^k u_n^k)$, one can approximate, in the metric of the complex space $\overset{\circ}{W}{}_2'(\Omega) \times \overset{\circ}{W}{}_2'(\Omega)$, any pair of functions (u, v) belonging to this space. Indeed, there exist $n(n-1)/2$ linearly independent monomials of the form $(z\bar{z} - 1)z^l\bar{z}^k$, where $l + k \leq n - 2$.

Let M_n be the closed linear span of the vectors $((z\bar{z} - 1)z^l\bar{z}^k,\ (z\bar{z} - 1)z^m\bar{z}^j)$, where $l + k \leq n - 2,\ m + j \leq n - 2$, and let H_n be the linear span of the vectors $(u_\rho^k, \lambda_\rho^k u_\rho^k)$ for $\rho \leq n$. Since u_ρ^k are polynomials of degree ρ, in order to prove completeness it suffices to prove that the dimensions of the spaces M_n and H_n coincide. The latter assertion is obvious if the vectors $(u_\rho^k, \lambda_\rho^k u_\rho^k)$ are linearly independent. Suppose this is not so. Then there exists a system of numbers C_ρ^k such that

$$\sum C_\rho^k u_\rho^k = \sum \lambda_\rho^k C_\rho^k u_\rho^k = 0,$$

and the solution of the mixed problem for equation (1), constructed from zero initial data,

$$u = \sum C_\rho^k u_\rho^k e^{\lambda_\rho{}^k t}.$$

By virtue of the uniqueness of the solution of the indicated problem, $u = 0$. Hence it follows that $C_\rho^k = 0$, since the u_ρ^k corresponding to different λ_ρ^k are linearly independent; and if $u^k_{\rho_1}, u^k_{\rho_2}$ correspond to one and the same $\lambda^k_{\rho_1}$, then the degrees of these polynomials are different.

It is easy to see that if, for some λ, there exists a solution of equation (5) vanishing on Γ, then it has the form (7), and for any function Ψ_1

$$u = \Psi_1(\varphi(x + \mu_1 y)) - \Psi_1(-\psi(x + \mu_2 y)) \tag{16}$$

satisfies the same equation and the zero boundary conditions.

Obviously, there are infinitely many linearly independent functions of the form (16). Thus, either the solution of the Dirichlet problem for equation (5) is unique, or there exist infinitely many linearly independent solutions. We note that the eigenvalues obtained by us are situated everywhere densely on the curve $|4\lambda^2 + 1| = 1$ in the complex λ-plane. Equation (5) for $\lambda = 0$ is parabolic; for all other values of λ lying on this curve, the roots of the characteristic polynomial are complex and distinct.

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Note: Figure translations are in progress. See original paper for figures.

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