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Abstract

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THEORY OF ELASTICITY

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ON THE SOLUTION OF PROBLEMS FOR A SHALLOW SPHERICAL SHELL IN THE CASE OF MULTIPLY CONNECTED DOMAINS

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In the work ([1]), for the study of the stress state in a shallow spherical shell in the case of multiply connected domains, a method of successive approximations was proposed; moreover, the authors restricted themselves only to the first approximation, which made it possible to determine at what distance the holes do not affect one another. Here a procedure is proposed for reducing problems on the stress state of shallow spherical shells in the case of multiply connected domains to infinite systems of algebraic equations.

§ 1. Consider the stress state of a shallow spherical shell which, in the plane, occupies an $(m+1)$-connected domain S, bounded by the circle L_0 (Fig. 1). L_k are circles with centers to which the coordinate systems (x_k, y_k) are attached; the coordinate system (x, y) is attached to L_0.

The study of the stress state reduces to the solution of the equation

$$\nabla^2(\nabla^2 - i\varkappa^2)\Phi = \frac{qr_0^4}{D} \tag{1}$$

under the boundary conditions

$$\mathcal{L}_k^{(t)}\Phi\big|_{L_k} = f_{kz}(\theta_k); \tag{2}$$

$$t = 1, 2, 3, 4; \quad z_k = x_k + iy_k; \quad z_k = r_k e^{i\theta_k}; \quad k = 0, 1, \ldots, m; \quad z = z_0; \quad z = re^{i\theta},$$

where $\Phi = w + ig\varphi$; D is the cylindrical stiffness; q is the intensity of the normal load; all coordinates are dimensionless, referred to r_0, and are related by the relations $z = z_k + l_k$; $g = \sqrt{12(1-\nu^2)/E^2h^4}$; $\varkappa = r_0 \sqrt[4]{12(1-\nu^2)/R^2h^2}$; $\mathcal{L}_k^{(t)}$ are the differential operators of the boundary conditions.

The solution of equation (1) will be written in the form of the sum

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3; \tag{3}$$

Φ_1 is a solution of the Laplace equation, Φ_2 is a solution of the Helmholtz equation, and Φ_3 is a particular solution.

$$\Phi_1 = i \sum_{k=1}^{m} B_k \ln(z - l_k)(\bar{z} - \bar{l}_k) + \varphi(z) + \overline{\psi(z)}; \quad \operatorname{Im} B_k = 0, \tag{4}$$

where $\varphi(z)$ and $\psi(z)$ are functions holomorphic in S, which, following ([2]), we represent in the form

$$\varphi(z) = \sum_{k=1}^{m} \sum_{p=1}^{\infty} \frac{\alpha_{kp}}{(z-l_k)^p} + \sum_{p=0}^{\infty} \beta_p z^p; \quad \psi(z) = \sum_{k=1}^{m} \sum_{p=1}^{\infty} \frac{\alpha^*_{kp}}{(z-l_k)^p} + \sum_{p=0}^{\infty} \beta^*_p z^p. \tag{5}$$

Fig. 1

To determine the components of the stress and deformation states corresponding to Φ_1, in the coordinate system (r_μ, θ_μ), the relations obtained are:

$$S^1_{r_\mu \theta_\mu} + iT^1_{r_\mu} = -\frac{1}{gr_0^2} z_\mu \frac{\varphi''(z) - \psi''(z)}{\bar{z}_\mu} + \frac{2i}{gr_0^2} \sum_{k=1}^{m} \frac{B_k z_\mu}{(z-l_k)^2 \bar{z}_\mu}; \qquad T^1_{\theta_\mu} = -T^1_{r_\mu};$$

$$G^1_{r_\mu} = -D\frac{1-\nu}{r_0^2} \operatorname{Re} z_\mu \frac{\varphi''(z) + \psi''(z)}{\bar{z}_\mu}; \qquad G^1_{\theta_\mu} = -G^1_{r_\mu};$$

$$\widetilde{Q}^1_{r_\mu} = -\frac{2}{r_0 r_\mu} G^1_{r_\mu} + D\frac{1-\nu}{r_0^3 r_\mu} \operatorname{Re} z_\mu^2 \frac{\varphi'''(z) + \psi'''(z)}{\bar{z}_\mu}; \tag{6}$$

$$u^1_\mu + iv^1_\mu = -\frac{1+\nu}{Ehgr_0} \left[\overline{\varphi'(z)} - \overline{\psi'(z)} - 2i \sum_{k=1}^{m} \frac{B_k}{z-l_k} \right] \frac{\bar{z}_\mu}{r_\mu} - \frac{r_0}{R} \frac{\bar{z}_\mu}{r_\mu} \int [\varphi(z) + \psi(z)]\, dz + i\frac{r_0}{R} \frac{\bar{z}_\mu}{r_\mu} z \operatorname{Im}(\beta_0 + \beta^*_0).$$

We represent the solution of the Helmholtz equation for the domain S in the form

Fig. 2

Figure 1: Fig. 2

$$\Phi_2 = \sum_{k=1}^{m}\sum_{p=0}^{\infty}\begin{pmatrix} a_{kp} + ib_{kp} \\ a^*_{kp} + ib^*_{kp}\end{pmatrix} H_p^{(1)}(r_k\chi\sqrt{-i})\frac{\cos p\theta_k}{\sin p\theta_k} + \sum_{p=0}^{\infty}\begin{pmatrix} a_p + ib_p \\ a^*_p + ib^*_p\end{pmatrix} J_p(r\chi\sqrt{-i})\frac{\cos p\theta}{\sin p\theta}; \tag{7}$$

$$H_p^{(1)}(r_k\chi\sqrt{-i}) = \operatorname{her}_p r_k\chi + i\ \operatorname{hei}_p r_k\chi; \qquad J_p(r\chi\sqrt{-i}) = \operatorname{ber}_p r\chi + i\ \operatorname{bei}_p r\chi.$$

The components of the stress and deformation states corresponding to Φ_2 and Φ_3, in the coordinate system (r_μ, θ_μ), are calculated by the usual formulas for the polar coordinate system (3). The displacements corresponding to Φ_2 satisfy the single-valuedness conditions for displacements; assuming that these conditions are also satisfied by the displacements corresponding to Φ_3, from the single-valuedness conditions for displacements, taking (6) into account, we obtain $\alpha_{k1} + \alpha^*_{k1} = 0$.

Fig. 2

Thus the components of the stress and deformation state are determined in any coordinate system (r_μ, θ_μ). Having calculated, from these components, the quantities entering into the boundary conditions (2), we expand them in Fourier series on the μ-th contour. When computing integrals containing the function Φ_1, one should pass to the domain of the complex variable (4), where they are computed rather easily by the residue theorem; and when computing integrals containing the function Φ_2, one must use the addition theorem for cylindrical functions (5). Equating to zero the coefficients of the harmonics on each of the contours, we obtain an infinite system of algebraic equations containing $4(m+1)$ rows of undetermined constants.

§ 2. As an example, let us write the infinite system of algebraic equations for the problem of the stress state in a spherical shell around two equal circular holes that are free in the sense of (6), of radius r_0, the distance between whose centers is equal to lr_0, under uniform internal pressure of intensity q. We shall assume the basic stress state to be momentless (6). The solution of this problem reduces to the solution in the infinite domain S (Fig. 2) of the homogeneous equation (1) ($\Phi_3 = 0$) under deter-

conditions "at infinity" ([6]) and the following boundary conditions:

$$T_{r_\mu}\big|_{r_\mu=1} = -p_0 h; \qquad S_{r_\mu\theta_\mu}\big|_{r_\mu=1} = 0; \qquad G_{r_\mu}\big|_{r_\mu=1} = 0; \qquad \widetilde{Q}_{r_\mu}\big|_{r_\mu=1} = -\frac{qr_0}{2};$$

$$p_0 = \frac{qR}{2h}; \qquad \mu = 1, 2. \tag{8}$$

By virtue of the force and geometric symmetry, Φ_1 and Φ_2 have the form

$$\Phi_1(z) = \varphi(z) + \overline{\psi(z)}; \qquad \psi(z) = \overline{\varphi(z)};$$

$$\varphi(z) = \sum_{p=1}^{\infty} \alpha_p \left[\left(z + \frac{l}{2} \right)^{-p} + (-1)^{-p} \left(z - \frac{l}{2} \right)^{-p} \right], \tag{9}$$

$$\Phi_2 = \sum_{p=0}^{\infty} (a_p + ib_p) \left[H_p^{(1)}(r_1 \chi \sqrt{-i}) \cos p\theta_1 + (-1)^p H_p^{(1)}(r_2 \chi \sqrt{-i}) \cos p\theta_2 \right].$$

From the conditions of single-valuedness of the displacements, $\operatorname{Re} \alpha_1 = 0$. We note that Φ_1 and Φ_2 in (9) satisfy the conditions "at infinity" ([6]). Let us write out the infinite system for determining a_p, b_p, c_p, and d_p ($\alpha_p = c_p + id_p$):

$$c_0 = 0; \qquad d_0 = 0; \qquad c_1 = 0; \qquad C_n^1 = 0; \qquad C_n^2 = 0; \qquad C_{np}^1 = 0; \qquad C_{np}^2 = 0;$$

$$D_n^3 = 0; \qquad D_n^4 = 0; \qquad D_{np}^3 = 0; \qquad D_{np}^4 = 0;$$

$$A_n^t a_n + B_n^t b_n + C_n^t c_n + D_n^t d_n + \sum_{p=0}^{\infty} \left(A_{np}^t a_p + B_{np}^t b_p + C_{np}^t c_p + D_{np}^t d_p \right)$$

$$= -p_0 \frac{R}{E} \delta_n^0 \left(\delta_t^1 + \chi \delta_t^4 \right); \tag{10}$$

$$t = 1, 2, 3, 4; \qquad n = 0, 1, \ldots, \infty; \qquad \delta_n^k = \begin{cases} 1, & n = k, \\ 0, & n \neq k; \end{cases} \qquad \varepsilon_n = \begin{cases} 1/2, & n = 0, \\ 1, & n \neq 0. \end{cases}$$

Here the following notation has been introduced:

$$A_n^1 = \operatorname{her}_n \chi - \operatorname{hei}_n' \chi; \qquad A_n^2 = n(\chi \operatorname{hei}_n' \chi - \operatorname{hei}_n \chi);$$

$$A_n^3 = (1-\nu)\,\mathrm{her}_n''\,\chi - \nu\,\mathrm{hei}_n\,\chi; \qquad A_n^4 = \mathrm{hei}_n'\,\chi + n^2\frac{1-\nu}{\chi^2}\left(\chi\,\mathrm{her}_n'\,\chi - \mathrm{her}_n\,\chi\right);$$

$$B_n^1 = -\,\mathrm{hei}_n\,\chi - \mathrm{her}_n'\,\chi; \qquad B_n^2 = n(\chi\,\mathrm{her}_n'\,\chi - \mathrm{her}_n\,\chi);$$

$$B_n^3 = -(1-\nu)\,\mathrm{hei}_n''\,\chi - \nu\,\mathrm{her}_n\,\chi; \qquad B_n^4 = \mathrm{her}_n'\,\chi - n^2\frac{1-\nu}{\chi^3}\left(\chi\,\mathrm{hei}_n'\,\chi - \mathrm{hei}_n\,\chi\right);$$

$$C_n^3 = 2\frac{1-\nu}{\chi^2}\,n(n+1); \qquad C_n^4 = -2\frac{1-\nu}{\chi^3}\,n^2(n+1);$$

$$D_n^1 = -2\frac{n(n+1)}{\chi^2}; \qquad D_n^2 = -2n(n+1);$$

$$A_{np}^1 = \varepsilon_n\{[\mathrm{her}_{p+n}\,l\chi + (-1)^n\,\mathrm{her}_{p-n}\,l\chi](\mathrm{ber}_n\,\chi - \mathrm{bei}_n'\,\chi)$$

$$-[\mathrm{hei}_{p+n}\,l\chi + (-1)^n\,\mathrm{hei}_{p-n}\,l\chi](\mathrm{bei}_n\,\chi + \mathrm{ber}_n''\,\chi)\}; \qquad (11)$$

$$A_{np}^2 = n\varepsilon_n\{[\mathrm{her}_{p+n}\,l\chi + (-1)^n\,\mathrm{her}_{p-n}\,l\chi](\chi\,\mathrm{bei}_n'\,\chi - \mathrm{bei}_n\,\chi)$$

$$+[\mathrm{hei}_{p+n}\,l\chi + (-1)^n\,\mathrm{hei}_{p-n}\,l\chi](\chi\,\mathrm{ber}_n'\,\chi - \mathrm{ber}_n\,\chi)\};$$

$$A_{np}^3 = \varepsilon_n\{[\mathrm{her}_{p+n}\,l\chi + (-1)^n\,\mathrm{her}_{p-n}\,l\chi][(1-\nu)\,\mathrm{ber}_n''\,\chi - \nu\,\mathrm{bei}_n\,\chi]$$

$$-[\mathrm{hei}_{p+n}\,l\chi + (-1)^n\,\mathrm{hei}_{p-n}\,l\chi][(1-\nu)\,\mathrm{bei}_n''\,\chi + \nu\,\mathrm{ber}_n\,\chi]\};$$

$$A^4_{np} = \varepsilon_n\Big\{[\operatorname{her}_{p+n} l\chi + (-1)^n \operatorname{her}_{p-n} l\chi]\Big[\operatorname{bei}'_n \chi + n^2\frac{1-\nu}{\chi^3}(\chi \operatorname{ber}'_n \chi - \operatorname{ber}_n \chi)\Big] +$$
$$+ [\operatorname{hei}_{p+n} l\chi + (-1)^n \operatorname{hei}_{p-n} l\chi]\Big[\operatorname{ber}'_n \chi - n^2\frac{1-\nu}{\chi^3}(\chi \operatorname{bei}'_n \chi - \operatorname{bei}_n \chi)\Big]\Big\};$$
$$B^1_{np} = -\varepsilon_n\Big\{[\operatorname{her}_{p+n} l\chi + (-1)^n \operatorname{her}_{p-n} l\chi](\operatorname{bei}_n \chi + \operatorname{ber}''_n \chi) +$$
$$+ [\operatorname{hei}_{p+n} l\chi + (-1)^n \operatorname{hei}_{p-n} l\chi](\operatorname{ber}_n \chi - \operatorname{bei}''_n \chi)\Big\};$$
$$B^2_{np} = n\varepsilon_n\Big\{[\operatorname{her}_{p+n} l\chi + (-1)^n \operatorname{her}_{p-n} l\chi](\chi \operatorname{ber}'_n \chi - \operatorname{ber}_n \chi) -$$
$$- [\operatorname{hei}_{p+n} l\chi + (-1)^n \operatorname{hei}_{p-n} l\chi](\chi \operatorname{bei}'_n \chi - \operatorname{bei}_n \chi)\Big\};$$
$$B^3_{np} = -\varepsilon_n\Big\{[\operatorname{her}_{p+n} l\chi + (-1)^n \operatorname{her}_{p-n} l\chi][(1-\nu) \operatorname{bei}''_n \chi + \nu \operatorname{ber}_n \chi] +$$
$$+ [\operatorname{hei}_{p+n} l\chi + (-1)^n \operatorname{hei}_{p-n} l\chi][(1-\nu) \operatorname{ber}''_n \chi - \nu \operatorname{bei}_n \chi]\Big\};$$
$$B^4_{np} = \varepsilon_n\Big\{[\operatorname{her}_{p+n} l\chi + (-1)^n \operatorname{her}_{p-n} l\chi]\Big[\operatorname{ber}'_n \chi - n^2\frac{1-\nu}{\chi^3}(\chi \operatorname{bei}'_n \chi - \operatorname{bei}_n \chi)\Big] -$$
$$- [\operatorname{hei}_{p+n} l\chi + (-1)^n \operatorname{hei}_{p-n} l\chi]\Big[\operatorname{bei}'_n \chi + n^2\frac{1-\nu}{\chi^3}(\chi \operatorname{ber}'_n \chi - \operatorname{ber}_n \chi)\Big]\Big\};$$
$$C^3_{np} = 2\frac{1-\nu}{\chi^2} l^{-p-n}\frac{(n+p-1)!}{(n-2)!(p-1)!}; \qquad C^4_{np} = 2\frac{1-\nu}{\chi^3} n l^{-p-n}\frac{(n+p-1)!}{(n-2)!(p-1)!};$$
$$D^1_{np} = -2\frac{l^{-p-n}}{\chi^2}\frac{(n+p-1)!}{(n-2)!(p-1)!}; \qquad D^2_{np} = 2l^{-p-n}\frac{(n+p-1)!}{(n-2)!(p-1)!}.$$

In system (10), for $n = 0$ the second equation ($t = 2$) drops out, while the first and fourth ($t = 1, 4$) coincide; for $n = 1$ the first and second equations ($t = 1, 2$) also coincide. Under these conditions, system (10) gives the solution of the stated problem.

Restricting ourselves to a finite number of equations in system (10), we obtain an approximate solution in any approximation.

In the case of the plane problem, systems of equations for several approximations were obtained by a similar method in (7). It turned out that the discrepancy in the first approximation, as compared with the practically exact solution, is small even when the distance between the holes is equal to half the radius of a hole.

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Note: Figure translations are in progress. See original paper for figures.