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Abstract

Full Text

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On Factor Mappings of Metric Spaces

(Presented by Academician P. S. Aleksandrov, 26 XI 1963)

This work consists of two parts, of which the first is devoted to the problem of preservation of metrizability of a space under various special types of factor mappings. In the second, the notion of a covering mapping is introduced and its place is clarified. General, in particular historical, information relevant to the material of the paper may be drawn from the detailed and modern surveys of P. S. Aleksandrov ([1,2]). Let us note that all spaces are assumed to be completely regular, and all mappings continuous. Proofs are omitted throughout. The terminology is the same as in ([4–6]).

I. § 1. In ([6]) I formulated the following result: the factor space of a metric space is metrizable if and only if the corresponding factor mapping is regular and pseudo-open. The question arose: is not regularity alone of the factor mapping under consideration sufficient here? The answer is not known to me, but I think that it is negative. Nevertheless, in a number of cases one can obtain a positive result.

By $[M]_{k_1}$, where $M \subseteq X$, X is a topological space, we agree to denote the set of all those points $x \in X$ for which there exists a bicompactum $F \subseteq X$ such that $x \in [F \cap M]$. Suppose that for $M \subseteq X$ it has already been defined what we mean by $[M]_{k_n}$, where n is some integer. Then put $[M]_{k_{n+1}} = [[M]_{k_n}]_{k_1}$.

Definition 1. A space X is called a k_n**-space** ($n > 0$ is a fixed integer) if for every set $M \subseteq X$

$$[M]_{k_n} = [M].$$

Theorem 1. *The factor space of a metrizable space is metrizable if and only if, for some integer $n > 0$, it is a k_n-space and the mapping under consideration is regular.*

Since spaces that are complete in the sense of Čech, like any feathered spaces ([5]), are spaces of point-countable type ([5]), they are also k_2-spaces. Therefore we have:

Corollary 1. *The factor space of a metrizable space is metrizable if and only if it is feathered and the corresponding mapping is regular.*

In particular:

Corollary 2. *A bicompactum that is a factor regular image of a metrizable space is metrizable.*

Problem. Whether an analogue of Corollary 2 is true for an arbitrary final-compact space is not yet known.

§ 2. A broader class, in comparison with regular mappings, is formed by the uniform mappings of metric spaces introduced in ([7]) by V. I. Ponomarev; these include, in particular, all bicompact mappings (see ([7])).

Ponomarev obtained the first interesting result on uniform mappings: a paracompactum that is an open uniform image

of a metric space, is metrizable. Later I noticed that metrizability is preserved under arbitrary closed uniform mappings, and without any additional restrictions on the image. The question remains open: will a paracompact space that is a pseudo-open uniform image of a metric space be metrizable? But recently a number of positive results have also been obtained.

Definition 2. A space is called **weakly finally compact** if every infinite set in it has a limit point*.

Recall that a mapping $f : X \to Y$ is called an s-mapping if, for every point $y \in Y$, the set $f^{-1}y$ is a space with a countable base.

Theorem 2. Let $f : X \to Y$ be a pseudo-open uniform s-mapping of a metric space X onto a space Y, satisfying one of the following conditions: a) Y is a weakly finally compact space complete in the sense of Čech, for example bicompact; b) $Y \subset B$, where B is some completely normal bicompact space; c) Y is the preimage of some metric space under some perfect mapping.

Then Y is a space with a countable base (and, consequently, metrizable).

For spaces with a countable base a stronger result is true.

Theorem 3. Under a pseudo-open uniform mapping, a space with a countable base is carried into a space with a countable base.

Corollary 3. A pseudo-open bicompact image of a metric space with a countable base is a space with a countable base.

Remark. The result of Theorem 3 is, to a certain extent, definitive; indeed, every mapping of a space X with a countable base onto a regular space Y with a countable base is, a fortiori, uniform with respect to some metric on X ([4]), i.e. uniformity of the mapping is a necessary condition for preservation of metrizability. From this it is also clear how much broader the class of uniform mappings is than the class of bicompact mappings.

For bicompact mappings, generally speaking, a stronger result than Theorem 2 holds:

Theorem 4. Let $f : X \to Y$ be a pseudo-open bicompact mapping of a metric space X onto a weakly finally compact space Y.

Then Y is a space with a countable base.

§ 3. In (4) the following theorem is given:

Among T_1-spaces, the spaces with a uniform base (3), and only they, are open bicompact images of strongly paracompact metric spaces.

But what if one additionally imposes on the mapping the requirement of monotonicity (i.e. connectedness of the preimages of points)?

Theorem 5. A space that is a pseudo-open, bicompact, monotone image of a strongly paracompact metric space is itself metrizable.

Let us note that here the requirement of bicompactness of the mapping can be weakened to the requirement of uniformity.

Problem. Is it true that a regular space which is an open, bicompact, monotone image of a metric space is metrizable?

§ 4. Here we shall consider arbitrary quotient bicompact mappings of metric spaces, i.e. the spaces of arbitrary decompositions of them into compacta. We shall indicate the solution of two problems: 1) when, under such mappings, is the image metrizable? and 2) when is such a mapping pseudo-open?

Theorem 6. Let $f : X \to Y$ be a quotient bicompact mapping of a metric space X onto a space Y, satisfying

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* That is, such a point every neighborhood of which contains infinitely many points of our set.

one of the following conditions: a) Y is a weakly finally compact paracompact space; b) $Y \subseteq B$, where B is some perfectly normal bicompactum; c) the space Y is a perfect image of some metric space with a countable base; d) Y is a finally compact space complete in the sense of Čech; e) Y is a bicompactum.

It seems to me that conditions c), d), and e) deserve to be singled out, although condition a) is more general with respect to them.

Theorem 7. A quotient bicompact mapping of a metric space is pseudo-open if and only if the image is a k_1-space.

Theorem 8. The decomposition space of a metric space with a countable base into bicompacta is metrizable if and only if it is a k_1-space.

§ 5. The requirements of bicompactness or uniformity of mappings that occur in the formulations of all the theorems given above cannot be discarded: by B. Ponomarev' s theorem, open images of metric spaces are all spaces with the first axiom of countability–therefore among them there are as many nonmetrizable spaces as desired, even nonmetrizable bicompacta. Nevertheless, with the aid of one of my results and one theorem of B. Efimov, one can prove the following:

Theorem 9. A dyadic bicompactum that is a quotient space of a metric space is metrizable.

II. § 6. This paragraph is devoted entirely to the question: how do the bicompact subsets of spaces behave under various kinds of quotient mappings?

Definition 3. A system φ of bicompact subsets of a space Y is called its **determining k-system** if a set $M \subseteq Y$ is closed whenever its intersection with every element of φ is closed.

Definition 4. A **determining k_1-system** of a space Y is a system ψ of its bicompact subsets such that, for any $M \subseteq Y$, $x \in [M]$ only in the case when, for some $\Phi \in \psi$, one has $x \in [\Phi \cap M]$.

Determining k-systems, respectively determining k_1-systems, exist only in k-spaces, respectively in k_1-spaces (in these cases, as examples of such systems one may choose the collections of all bicompact subsets of the spaces under consideration).

Theorem 10. A mapping $f : X \to Y$, where Y is a k-space, is quotient if and only if the images of all bicompacta lying in X form a determining k-system in Y.

Theorem 11. A mapping of a space X onto a k_1-space Y is pseudo-open if and only if the images of all bicompacta from X form a determining k_1-system in Y.

Definitions 3 and 4, and Theorems 10 and 11, suggest considering the following class of mappings:

Definition 5. A mapping $f : X \to Y$ is called **covering** if for every bicompactum $\Phi \subseteq Y$ there exists a bicompactum $F \subseteq X$ such that $fF \supseteq \Phi$.*

Trivial examples of covering mappings are perfect mappings.

Let $f : X \to Y$ be some mapping, let $C(X)$, $C(Y)$ be the spaces of continuous functions on X and Y, endowed with the bicompact-open topology, and let $f^* : C(Y) \to C(X)$ be the contravariant mapping generated by f.

Theorem 12. A mapping $f : X \to Y$ is covering if and only if $f^* : C(Y) \to f^*C(Y) \subseteq C(X)$ is a homeomorphism.

* Independently of me and for another reason, D. B. Fuks arrived at the concept of a covering mapping.

The following results, along with the principal theorem (12) stated above, speak to the role and breadth of the class of covering mappings.

Theorem 13. *Every covering mapping onto a k-space is factor, and onto a k_1-space is pseudo-open.*

This follows from Definition 5 and Theorems 10 and 11.

Theorem 14. *Every open mapping of a space complete in the sense of Čech is covering.*

Corollary 4. *An open mapping of an arbitrary complete metric space is covering.*

Remark. For arbitrary metric spaces, Corollary 4 is not true.

Theorem 15. *A closed mapping of any metric space is covering.*

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Note: Figure translations are in progress. See original paper for figures.