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### Abstract

### Full Text

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# ON SOME DIFFERENTIAL GAMES

## 1. Formulation of the problem

It is assumed that the state of an object is determined by a point $z = (z^1, \dots, z^n)$ of the vector $n$-dimensional space $R$, and its behavior is determined by a system of ordinary differential equations:

$$dz/dt = Z(z, u, v) = X(z, u) + Y(z, v), \tag{1}$$

whose right-hand sides are analytic functions; $u$ and $v$ are control parameters. Here $u$ is a point of an analytic $p$-dimensional manifold $P$, and $v$ is a point of an analytic $q$-dimensional manifold $Q$. In the space $R$ an analytic manifold $M$ of some dimension is given. The game is considered finished when the point $z$ reaches the manifold $M$. The problem consists in determining, at each instant of time, the behavior of the parameter $u$ that leads to completion of the game in the shortest time, knowing the state $z$ of the object at that instant of time and the value of the parameter $v$ at the same instant of time. It should be noted that at some points $z$ one must use not only the values of the parameter $v$ themselves, but also a certain number of its derivatives with respect to time. The parameter $v$ is assumed to be a piecewise-analytic function of time $t$.

## 2. Main result

As in the theory of optimal processes ([1]), alongside the contravariant vector $z$ we introduce a covariant vector $\psi = (\psi_1, \dots, \psi_n)$ and define the function $H$ by setting:

$$H(z, \psi, u, v) = \psi Z = \sum_{i=1}^{n} \psi_i Z^i(z, u, v). \tag{2}$$

For fixed values of the vectors $z$ and $\psi$, let us find the maximum $M(z, \psi)$ of the function $\psi X(z, u)$ and the minimum $m(z, \psi)$ of the function $\psi Y(z, v)$. We form the system of ordinary differential equations:

$$dz^i/ds^1 = \partial H/\partial \psi_i; \qquad d\psi_i/ds^1 = -\partial H/\partial z^i, \tag{3}$$

taking $s^1$ as the independent variable, and supplement this system with the finite relations

$$\begin{aligned} \psi X(z,u) &= M(z,\psi), \\ \psi Y(z,v) &= m(z,\psi). \end{aligned} \tag{4}$$

We shall assume that the system (3), (4) is solvable in the following sense. Let $z_0$ be an arbitrary point of $M$, and let $\psi^0$ be a unit covariant vector defining a hyperplane tangent to $M$ at the point $z_0$. We shall assume that the system (3), (4) has a unique solution

$$z = z(s^1); \qquad \psi = \psi(s^1); \qquad u = u(s^1); \qquad v = v(s^1), \tag{5}$$

defined for all values $s^1 \leq 0$ and satisfying the initial conditions $z(0) = z_0$; $\psi(0) = \psi^0$. The solution (5) depends on the initial pair $(z_0, \psi^0) = x$, and the totality of all such pairs forms an analytic manifold $N$ of dimension $n-1$, in which we introduce local coordi-

coordinates $s^2, \dots, s^n$. Forming all possible pairs of the form $(s^1, x)$, where $s^1$ is a negative number and $x$ is a point of the manifold $N$, we obtain a manifold $S$ of dimension $n$, whose points we shall denote by

$$s = (s^1, x) = (s^1, s^2, \dots, s^n).$$

Taking into account the dependence of the solution (5) on the initial conditions, we can write:

$$z = z(s) = \omega(s); \qquad \psi = \psi(s); \qquad u = \mathbf{u}(s); \qquad v = \mathbf{v}(s). \tag{6}$$

The function $\omega$ gives an analytic mapping of the manifold $S$ into the space $R$. In the case when the mapping $\omega$ is one-to-one and has a functional determinant nowhere vanishing, the problem was solved by Bellman. However, in very simple cases of interest the mapping $\omega$ is not one-to-one. The present paper is devoted to overcoming this difficulty under certain simple assumptions.

Among all points $s = (s^1, x)$ that pass into one and the same point $z$ under the mapping $\omega$, choose that for which the number $s^1$ has the greatest value. We shall say of this point $s$ that it belongs to the **upper layer**, and shall denote it by $\omega^{-1}(z)$.

**Theorem.** Let $\hat{z}$ be some point of $R$, and let

$$s_0 = (s_0^1, x_0) = \omega^{-1}(\hat{z}).$$

Then, starting from the state $\hat{z}$ of the object, the game can always be ended in a time not exceeding the number $|s_0^1|$.

This theorem is, of course, not true for an arbitrary game (1). Here it will be proved only under certain very restrictive assumptions.

Let $z(t)$ be a solution of equation (1), starting at $\hat{z}$ and ending on $M$. Put

$$\omega^{-1}(z(t)) = (s^1(t), x(t)).$$

Suppose that

$$\frac{ds^1(t)}{dt} \geq 1.$$

Then the game ends in a time not exceeding the number $|s_0^1|$. Consequently, it is sufficient for us to construct the control $u(t)$ as the control $v(t)$ becomes known, and to construct it in such a way that the inequality

$$\frac{ds^1}{dt} \geq 1$$

is satisfied at all times.

The controls

$$\tilde{u}(t) = \mathbf{u}(s_0^1 + t, x_0), \qquad \tilde{v}(t) = \mathbf{v}(s_0^1 + t, x_0)$$

are called **extremal**. They correspond to the extremal motion of the object

$$z = \omega(s_0^1 + t, x_0),$$

for which $ds^1/dt \equiv 1$.

Below we give some indications of the method of proof for a non-extremal control $v(t)$, as well as a formulation of the conditions under which the theorem is proved.

## 3. Recording equation (1) in the variables $s^1, s^2, \dots, s^n$

Put:

$$H(s,u,v) = H(z(s), \psi(s), u, v); \qquad H(s) = H(s, \mathbf{u}(s), \mathbf{v}(s)); \tag{7}$$

$$\delta H = \delta H(s,u,v) = H(s,u,v) - H(s). \tag{8}$$

From condition (4) it follows that:

$$H(s,u,\mathbf{v}(s)) \leq H(s); \qquad H(s,\mathbf{u}(s),v) \geq H(s). \tag{9}$$

In the case where the points $u$ and $v$ are respectively close to the points $\mathbf{u}(s)$ and $\mathbf{v}(s)$, one can give meaning to the quantities

$$\delta u = u - \mathbf{u}(s); \qquad \delta v = v - \mathbf{v}(s); \tag{10}$$

we shall regard them as vectors whose coordinates are computed in local coordinates of the manifolds $P$ and $Q$. Expanding the quantity $\delta H$ in a series in the coordinates of the vectors (10), we obtain

$$\delta H = -f_s(\delta u) + g_s(\delta v) + \cdots, \tag{11}$$

where $f_s$ and $g_s$ are nonnegative quadratic forms depending on $s$, and terms of order higher than the second have been omitted.

It is easily proved that

$$H(s) = \psi(s)\,\partial\omega(s)/\partial s^1; \qquad \partial H(s)/\partial s^1 = 0; \qquad \psi(s)\,\partial\omega(s)/\partial s^i = 0, \quad (12)$$

$$i = 2, \dots, n.$$

In what follows we shall assume that the following is satisfied.

**Condition 1.** At each point $s$ the vectors $\partial\omega(s)/\partial s^2, \dots, \partial\omega(s)/\partial s^n$ are linearly independent.

It follows from this that the functional determinant $D(s)$ of the mapping $\omega$ satisfies the condition $D(s) = d(s)H(s)$, where $d(s)$ does not vanish.

Let $\hat{z} = \omega(s_0)$, and let $s^1, \dots, s^n$ be local coordinates in a neighborhood of the point $s_0$. In order to write system (1) near the point $\hat{z}$ in the variables $s^1, \dots, s^n$, it is enough to solve the vector equation

$$Z(\omega(s), u, v) = \sum_{i=1}^{n} \frac{\partial\omega(s)}{\partial s^i}\frac{ds^i}{dt} \quad (13)$$

with respect to the quantities $ds^i/dt$. Multiplying relation (13) by $\psi(s)$ and dividing the result by $H(s)$, we obtain, by virtue of (12),

$$ds^1/dt = 1 + \delta H/H(s). \quad (14)$$

If $H(s_0) > 0$, then $D(s_0) = d(s_0)H(s_0) \neq 0$, and therefore equation (13) can be solved; in particular, relation (14) is valid. In this case, whatever the control $v(t)$, we define the control $u(t)$ by the relation $u(t) = u(s(t))$, and then, by virtue of (9), relation (14) gives $ds^1/dt \geq 1$. This corresponds to Bellman’ s solution.

If $H(s_0) = 0$, then we shall assume that the following is satisfied.

**Condition 2.** For $H(s_0) = 0$ we have $\operatorname{grad} H(s_0) \neq 0$.

Then there exists a vector $r(s) = (r^1(s), r^2(s), \dots, r^n(s))$, analytic in a neighborhood of the point $s_0$, such that $r^1(s) \equiv 1$ and

$$\sum_{i=1}^{n} \frac{\partial\omega(s)}{\partial s^i} r^i(s) = 0 \quad \text{when } H(s) = 0. \quad (15)$$

From (13), in addition to (14), one can derive that

$$\frac{ds^i}{dt} = \frac{\delta H}{H(s)}\, r^i(s) + R^i(s,u,v), \qquad \text{where } R^i(s,u(s),v(s)) = 0, \tag{16}$$

where $R^i(s,u,v)$ is an analytic function.

The derivative of a certain function $\varphi(s)$ by virtue of system (14), (16) is equal to

$$\frac{d\varphi(s)}{dt} = \frac{\delta H}{H(s)}\varphi_r(s) + \frac{\partial\varphi(s)}{\partial s^1} + \varphi_R(s,u,v), \tag{17}$$

where $\varphi_r(s) = \operatorname{grad}\varphi(s)\cdot r(s)$; $\varphi_R(s,u(s),v(s)) = 0$.

**4. Condition for solvability of the system (14), (16).** Let

$$s_0 = (s_0^1, x_0) = \omega^{-1}(\hat{z}), \qquad H(s_0) \geq 0.$$

Regarding the control $v(t)$, $t \geq 0$, as arbitrarily prescribed and non-extremal, we shall seek such a $u(t)$, $t \geq 0$, that the solution $s(t)$ of system (14), (16) with initial condition $s(0) = s_0$ satisfies the inequalities

$$ds^1(t)/dt > 1; \qquad H(s(t)) > 0 \quad \text{for } t > 0. \tag{18}$$

Since $v(t) \neq \tilde{v}(t)$, two cases are possible:

$$v(0) \neq v(s_0), \tag{19}$$

$$v(t) = \tilde{v}(t) + \tilde{b}t^m + O(t^{m+1}), \tag{20}$$

where $\tilde{b} \neq 0$ is some $q$-dimensional vector, and $m$ is a natural number.

In case (19) we impose two additional conditions.

**Condition 3.** $\delta H(s_0, u(s_0), v(0)) > 0$.

Put $H^0(s) = H(s),\ H^1(s) = H_r^0(s), \dots,\ H^{i+1}(s) = H_r^i(s)$ (see (17)).

**Condition 4.** For each point $s_0$ of the upper layer there exists a nonnegative integer $k$ such that

$$H^0(s_0) = 0, \dots,\ H^{k-1}(s_0) = 0, \quad H^k(s_0) > 0.$$

Putting $u(t) = u(s(t))$, under these conditions we can find a solution $s(t)$ of the system (14), (16) satisfying conditions (18).

In the case (20) we impose two further additional conditions.

**Condition 5.** The quadratic forms $f_{s_0}$ and $g_{s_0}$ (see (11)) are nondegenerate.

The quadratic forms with matrices inverse to the matrices of the forms $f_{s_0}$ and $g_{s_0}$ will be denoted by $\hat{f}_{s_0}, \hat{g}_{s_0}$. They are applicable to covariant vectors. Expand the function $H_R(s, u, v)$ (cf. (17)) in a Taylor series in $\delta u$ and $\delta v$:

$$H_R(s, u, v) = \lambda(s)\delta u + \mu(s)\delta v + \ldots$$

Here $\lambda(s)$ and $\mu(s)$ are covariant vectors.

**Condition 6.** $\hat{f}_{s_0}(\lambda(s_0)) > \hat{g}_{s_0}(\mu(s_0))$.

In the case (20) there exists a control $u(t)$ (generally speaking, not coinciding with $u(s(t))$) such that the solution $s(t)$ of the system (14), (16) satisfies conditions (18).

If, for $s^1$ close to $s^1_0$, the identities

$$\lambda(s^1, x_0) \equiv 0; \quad \mu(s^1, x_0) \equiv 0; \quad \frac{\partial}{\partial s^1} H_r(s^1, x_0) \equiv 0, \tag{21}$$

hold, then condition 6 is not fulfilled; but then, for $k = 2$ (see condition 4), for the control (20) one may take the control $u(t) = u(s(t))$, and the solution $s(t)$ will satisfy conditions (18).

**5. Example.** Let $a$ and $b$ be two objects whose geometric positions are determined by vectors $\xi$ and $\eta$ in a Euclidean space $E$ of arbitrary dimension. Their motions are given by the equations:

$$\ddot{\xi} + \alpha\dot{\xi} = \rho u, \qquad \ddot{\eta} + \beta\dot{\eta} = \sigma v.$$

Here $\alpha, \beta, \rho, \sigma$ are positive numbers; $u$ and $v$ are control vectors from $E$, of modulus equal to 1. The game consists in the pursuit of object $b$ by object $a$. The theory set out above is applicable to it if the inequalities

$$\rho > \sigma, \qquad \rho/\alpha > \sigma/\beta$$

are fulfilled.

In computing this example the following general proposition is used.

Let $z$ be an arbitrary point of $R$ not belonging to $M$. Suppose that the totality of all such negative numbers $s^1$ for which $\omega(s^1, x) = z$ is determined from the equation $F(s^1, z) = 0$, and that, for negative $s^1$, this equation is incompatible with the equations

$$\frac{\partial}{\partial s^1} F(s^1, z) = 0; \quad \frac{\partial}{\partial z^i} F(s^1, z) = 0, \quad i = 1, \ldots, n.$$

Then, if condition 1 is fulfilled, for negative $s^1$ the relation

$$H(s) = a(s)\frac{\partial}{\partial s^1} F(s^1, z),$$

holds, where $z = \omega(s)$, $s^1$ is a root of the equation $F(s^1, z) = 0$, and $a(s)$ does not vanish.

This proposition makes it possible to verify condition 4; namely, if $a(s) > 0$, condition 4 is equivalent to the condition

$$\frac{\partial}{\partial s^1} F(s_0^1, \hat{z}) = 0, \dots, \quad \frac{\partial^k}{(\partial s^1)^k} F(s_0^1, \hat{z}) = 0, \quad \frac{\partial^{k+1}}{(\partial s^1)^{k+1}} F(s_0^1, \hat{z}) > 0.$$

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## References

1. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, *Mathematical Theory of Optimal Processes*, Moscow, 1962.

*Note: Figure translations are in progress. See original paper for figures.*