V. B. OZHEGOV

1964

SovietRxiv

Abstract

Full Text

V. B. OZHEGOV

ON SOME EXTREMAL PROPERTIES OF GENERALIZED APPELL POLYNOMIALS

(Presented by Academician S. N. Bernstein, 12 VI 1964)

1°. A sequence of polynomials $\{P_n(x)\}_0^\infty$ of degree exactly n will be called a sequence of generalized Appell polynomials of class $A^{(k)}$ if

$$P_n^{(k)}(x) = P_{n-k}(x) \quad (n = k,\ k+1,\ k+2, \dots). \tag{1}$$

It follows from (1) that if $\{P_n(x)\}_0^\infty \in A^{(k)}$, then the sequences $\{P_{n,j}(x)\}_{n=0}^\infty$ $(j = 0, \dots, k-1)$, where

$$P_{n,0}(x) \equiv P_n(x), \qquad P_{n,j}(x) \equiv P_{n+j}^{(j)}(x) \tag{2}$$

$$(j = 1, \dots, k-1;\ n = 0, 1, 2, \dots),$$

also belong to the class $A^{(k)}$.

We shall also say (which is equivalent to definition (1)) that $\{P_n(x)\}_0^\infty \in A^{(k)}$ if there exist (formally) k power series (in what follows–generating functions)

$$A_m(t) = \sum_{l=0}^{\infty} a_{l,m} t^l \qquad (m = 0, \dots, k-1)$$

such that (also formally)

$$\sum_{m=0}^{k-1} A_m(t) e^{\varepsilon_m t x} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

where ε_m are the primitive k-th roots of unity.

Theorem 1. *In order that $\{P_n(x)\}_0^\infty \in A^{(k)}$, it is necessary and sufficient that the conditions*

$$\int_0^\infty P_n^{(m)}(x)\, d\gamma(x) = \alpha_{n,m},$$

hold, where $\gamma(x)$ *is a function of bounded variation on* $(0, \infty)$*, all moments* γ_n *of which exist and* $\gamma_0 \neq 0$*;* $\alpha_{n,m}$ $(m = 0, \dots, n;\ n = 0, 1, 2, \dots)$ *are the elements of an infinite triangular table satisfying the condition*

$$\alpha_{n+k,m+k} = \alpha_{n,m} \quad (m = 0, \dots, n;\ n = 0, 1, 2, \dots).$$

This theorem generalizes results of C. J. Thorne (^{1}) for sequences of Appell polynomials (class $A^{(1)}$).

Corollary 1. *The generating functions of a sequence* $\{P_n(x)\}_0^\infty \in A^{(k)}$ *have the form*

$$A_m(t) = \frac{\sum_{n=0}^{\infty} \left[\sum_{l=0}^{k-1} \Delta^{(k)}_{l+1,m+1} \alpha_{n+l,l}\right] t^n}{\Delta^{(k)} \sum_{n=0}^{\infty} \varepsilon_m^n \frac{\gamma_n}{n!} t^n},$$

where $\Delta^{(k)}$ is the Vandermonde determinant $W(\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{k-1})$; $\Delta^{(k)}_{i,j}$ are the algebraic complements of its elements.

For example, for the sequence $\{S_n(x)\}_0^\infty \in A^{(2)}$ of Euler–Bernstein polynomials ((^{2}, p. 497), the generating functions have the form

$$A_0(t) = \frac{1 + e^{-t}}{e^t + e^{-t}}, \qquad A_1(t) = \frac{e^t - 1}{e^t + e^{-t}},$$

i.e.,

$$\frac{\operatorname{sh} tx + \operatorname{ch} t(1 - x)}{\operatorname{ch} t} = \sum_{n=0}^{\infty} S_n(x) t^n.$$

Corollary 2. Any polynomial $P_n(x)$ from the sequence $\{P_n(x)\}_0^\infty \in A^{(k)}$ can be represented in the form

$$P_{mk+l}(x) = \sum_{s=0}^{k-1} \sum_{i=r}^{m} \frac{p^{(0)}_{ik+l-s,s}}{(mk - ki + s)!} x^{mk-ki+s}, \tag{3}$$

where $l = 0, \dots, k-1$; $m = 0, 1, 2, \dots$; $r = 0$ when $l - s \geq 0$; $r = 1$ when $l - s < 0$; $p^{(0)}_{m,j}$ $(j = 0, \dots, k-1)$ are the constant terms of the polynomials (2).

In what follows we shall assume that, for all n,

$$P_n^{(n)}(x) = 1. \tag{4}$$

2°. Following S. N. Bernstein (([2], p. 515), consider the class ${}^{(j)}_{\lambda,1}$ of regularly monotone polynomials on $[0,1]$ with type numbers $\lambda_1 = \lambda - j$, $\lambda_{2k} = 1$, $\lambda_{2k+1} = \lambda$. From the generalized theorem of S. N. Bernstein (([2], p. 515) it follows that, among all polynomials of the class under consideration satisfying (4), the one that deviates least from zero on $[0,1]$ is the one satisfying the conditions

$$P_n^{(i)}(0) = 0, \qquad i \not\equiv \lambda \pmod{\lambda + 1};$$

$$P_n^{(i)}(1) = 0, \qquad i \equiv \lambda \pmod{\lambda + 1}. \tag{5}$$

Consider the sequence $\{P_n(x)\}_0^\infty$ of extremal polynomials of the class ${}^{(0)}_{\lambda,1}$. By condition (5), each polynomial can be represented in the form of an Abel-Goncharov integral

$$P_n(x) = \int_{\alpha_0}^{x} dx_1 \int_{\alpha_1}^{x_1} dx_2 \int_{\alpha_2}^{x_2} dx_3 \cdots \int_{\alpha_{n-1}}^{x_{n-1}} dx_n,$$

where $\alpha_i = 0$ when $i \not\equiv \lambda \pmod{\lambda+1}$ and $\alpha_i = 1$ when $i \equiv \lambda \pmod{\lambda+1}$.

It follows from this representation that the polynomials under consideration satisfy (1) for $k = \lambda + 1$, i.e., $\{P_n(x)\}_0^\infty \in A^{(\lambda+1)}$, and for $P_n(x)$ the representation (3) is valid with $k = \lambda + 1$. Moreover, the polynomials of the sequences $\{P_{n,j}(x)\}_{n=0}^\infty$, defined by condition (2), also deviate least from zero among polynomials of the class ${}^{(j)}_{\lambda,1}$, and for them one can write a representation analogous to (3). However, representation (3) contains the constant terms of the polynomials of all the sequences $\{P_{n,j}(x)\}_{n=0}^\infty$ $(j = 0, \dots, k-1)$. From the conditions (5), however, it follows that, for the extremal sequences, the only constant terms different from zero will be those of the sequence $\{P_{n,\lambda}(x)\}_{n=0}^\infty$.

Thus, the following is true.

Theorem 2. Among all polynomials of degree n from the class ${}^{(j)}_{\lambda,1}$ of the form

$$P_n(x) = \pm\frac{x^n}{n!} + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n$$

deviates least from zero on $[0,1]$ is the polynomial $\pm P_{n,j}(x)$, where

$$P_{n,j}(x) = \frac{x^n}{n!} + \sum_{i=1}^{m} \frac{p^{(0)}_{(\lambda+1)i+l+j-\lambda,\lambda}}{[(\lambda+1)(m-i)+\lambda-l]!}\, x^{(\lambda+1)(m-i)+\lambda-j}$$

$$(j = 0, \lambda, \dots; \quad l = -j, \dots, -j+\lambda; \quad n = m + (\lambda+1) + l; \quad m = 1, 2, 3, \dots),$$

whose coefficients are determined from the conditions

$$\frac{1}{n!}+\sum_{i=1}^{m}\frac{p^{(0)}_{(\lambda+1)i+l,\lambda}}{[(\lambda+1)(m-i)]!}=0$$

$$(n=m(\lambda+1)+l;\quad l=-\lambda,\dots,0;\quad m=1,2,3,\dots),$$

and this least deviation is equal to

$$L_n^{(\lambda)}=|p^{(0)}_{n,\lambda}|,\qquad L_n^{(j)}=\left|\frac{1}{n!}+\sum_{i=1}^{m}\frac{p^{(0)}_{(\lambda+1)i+l+j-\lambda,\lambda}}{[(\lambda+1)(m-i)+\lambda-j]!}\right|$$

$$(j=0,\dots,\lambda-1).$$

Corollary. The generating functions of the sequence $\{P_n(x)\}_0^\infty$ of extremal polynomials of the class ${}^{(0)}_{\lambda,1}$ have the form:

$$A_0(t)=1+\frac{1-e^t}{\sum_{j=0}^{\lambda}\varepsilon_j t},\qquad A_m(t)=\varepsilon_m\frac{1-e^t}{\sum_{j=0}^{\lambda}\varepsilon_j t}\quad(m=1,\dots,\lambda).$$

For example, for the extremal sequence $\{P_n(x)\}_0^\infty\in{}^{(0)}_{2,1}$ ((^{2}), $p.515$), the generating functions have the form

$$A_0(t)=1+\psi(t),\qquad A_1(t)=\varepsilon_1\psi(t),\qquad A_2(t)=\varepsilon_2\psi(t),$$

where

$$\psi(t)=\frac{1-e^t}{e^t+2e^{-1/2t}\cos\frac{\sqrt{3}}{2}t}.$$

Consequently,

$$e^{tx}+(e^{tx}+\varepsilon_1e^{\varepsilon_1tx}+\varepsilon_2e^{\varepsilon_2tx})\psi(t)=\sum_{n=0}^{\infty}P_n(x)t^n,$$

whence

$$1+3\psi(t)=\sum_{n=0}^{\infty}p^{(0)}_{n,2}t^n.$$

Thus, the coefficients in the expansion of the function $1 + 3\psi(t)$ in powers of t, up to sign, coincide with the quantities $L_n^{(2)}$ of the deviations of polynomials of degree n of the class ${}^{(2)}_{2,1}$. Hence, in particular, there follows S. N. Bernstein’ s formula ((^{2}), p.516)

$$\sum_{n=0}^{\infty} p^{(0)}_{3n,2} t^{3n} = \frac{3}{e^t + 2e^{-1/2t} \cos \frac{\sqrt{3}}{2} t}.$$

Leningrad Mechanical Institute

Received
1 VI 1964

REFERENCES

1. C. J. Thorne, *Am. Math. Monthly*, **52**, 191 (1945).

2. S. N. Bernstein, *Collected Works*, 2, No. 100, Publishing House of the Academy of Sciences of the USSR, 1954.

Note: Figure translations are in progress. See original paper for figures.