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Abstract

Full Text

MATHEMATICS

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A NONRANDOMIZED HOMOGENEOUS TEST IN THE BEHRENS-FISHER PROBLEM

For two repeated normal samples:

$$x_1, \dots, x_{n_1} \in N(a_1, \sigma_1^2); \qquad y_1, \dots, y_{n_2} = N(a_2, \sigma_2^2)$$

the Behrens–Fisher problem of testing the hypothesis $H_0: \; a_1 = a_2$ is considered by means of tests $\phi(x_1, \dots, x_{n_1}; y_1, \dots, y_{n_2})$ similar with respect to the nuisance parameters. For the general problem of describing all such randomized tests, at present one can construct a system of integro-differential equations with peculiar boundary conditions. One can also construct many types of nonrandomized similar tests (among which the best known is the Bartlett–Scheffé test ([1])). In the present note we deal with the narrower, unsolved question of the existence of nonrandomized homogeneous similar tests:

$$\phi = \phi\left(\frac{\bar{x} - \bar{y}}{s_2}, \frac{s_1}{s_2}\right),$$

where $\bar{x}, \bar{y}, s_1, s_2$ are sufficient statistics of the problem in the usual notation. Attempts at a formal construction of such tests have not given convincing results (see the literature up to 1955 in the survey of G. Breni ([2])). A. Wald ([3]) constructed an approximately similar test of this type. If the boundary separating the region of values $\phi = \phi\left(\frac{\bar{x}-\bar{y}}{s_2}, \frac{s_1}{s_2}\right) = 1$ from $\phi = 0$ satisfies certain conditions of piecewise smoothness, then nonrandomized tests with a critical region of the form

$$\frac{\bar{x} - \bar{y}}{s_2} = \Phi\left(\frac{s_1}{s_2}\right)$$

do not exist (Yu. V. Linnik ([4,5])); concerning families of homogeneous tests see O. V. Shalaevskii ([6]). However, if only measurability is required of a nonrandomized homogeneous test ϕ, then, as it turns out, it exists for any level.

Theorem 1. For any level $\alpha \in (0,1)$ and pairs of samples of sizes n_1, n_2 of different parity, there exists a measurable nonrandomized similar test for the Behrens-Fisher problem with critical region determined by the values

$$\frac{|\bar{x}-\bar{y}|}{s_2} \quad \text{and} \quad \frac{s_1}{s_2}.$$

Theorem 1 can be somewhat strengthened:

Theorem 2. Suppose a finite number K of pairs of samples of sizes n_{1i}, n_{2i} ($i = 1, 2, \ldots, K$) of different parity is given. Then there exists a measurable nonrandomized homogeneous test

$$\phi = \phi\left(\frac{|\bar{x}-\bar{y}|}{s_2}, \frac{s_1}{s_2}\right),$$

which is similar simultaneously for all these pairs of samples and has the prescribed level.

We shall briefly outline the main steps in the construction of the test indicated in Theorem 1. Let a level $\alpha \in (0,1)$ be given. To construct the desired test ϕ, consider the "test complement" $\psi = \phi - \alpha$, a measurable function taking only the values $1-\alpha$ and $-\alpha$. Introduce the "critics" of the family of measures induced by our pair of samples:

$$A = A\left(\frac{|\bar{x}-\bar{y}|}{s_2}, \frac{s_1}{s_2}\right); \quad B = B\left(\frac{|\bar{x}-\bar{y}|}{s_2}, \frac{s_1}{s_2}\right)$$

(see (5, 7)). Then the question reduces to the construction of a measurable function $\psi(A,B)$, taking only the values $1-\alpha$ and $-\alpha$, such that:

$$J(\theta) = \iint_{\Pi} \frac{\psi(A,B)(AB)^{(n_1-3)/2}(B-A)\,dA\,dB}{\sqrt{1-A}\sqrt{B-1}(\theta+A)^N(\theta+B)^N} = 0 \tag{1}$$

for all $\theta > 0$. Here Π is the half-strip $0 \le A \le 1$; $1 \le B < \infty$; $N = (n_1 + n_2 - 1)/2$; in what follows we shall assume $n_2 > n_1 \ge 3$ (the latter condition is introduced here to simplify the reasoning).

We divide the half-strip Π into disjoint half-strips Π_k: $1 - 2^{-k} \le A < 1 - 2^{-k-1}$; $1 \le B < \infty$; $\Pi = \bigcup_{k=0}^{\infty} \Pi_k$. We shall construct the function $\psi(A,B)$ so that for every $k = 0, 1, 2, \ldots$ one has

$$J_k(\theta) = \iint_{\Pi_k} \frac{\psi(A,B)(AB)^{(n_1-3)/2}(B-A)\,dA\,dB}{\sqrt{1-A}\sqrt{B-1}(\theta+A)^N(\theta+B)^N} = 0 \tag{2}$$

for all values $\theta > 0$. Since the function $(B-A)/\sqrt{1-A}\sqrt{B-1}$ has only an integrable singularity at $(1,1)$ of our half-strip, and from the condition $n_2 > n_1 \geq 3$, the relations (2) for $k = 0, 1, 2, \ldots$ will imply (1) for all $\theta > 0$. We shall consider the left-hand side of (2) for some value $k \geq 0$. Since n_1 and n_2 have different parity, $N = (n_1 + n_2 - 1)/2$ is an integer, and the expression

$$\frac{1}{(\theta+A)^N(\theta+B)^N}$$

can be decomposed into simple rational fractions. As a result, for any bounded and measurable $\psi(A,B)$ we find:

$$J_k(\theta) = \sum_{m=1}^{N} D_m \int_{1-2^{-k}}^{1-2^{-k-1}} dA \int_1^\infty dB \, \frac{\psi(A,B)(AB)^{(n_1-3)/2}(B-A)^m}{\sqrt{1-A}\sqrt{B-1}(B-A)^{2N-1}(\theta+A)^m} + \sum_{m=1}^{N} E_m \int_1^\infty dB \int_{1-2^{-k}}^{1-2^{-k-1}} dA \, \frac{\psi(A,B)(AB)^{(n_1-3)/2}(B-A)^m}{\sqrt{1-A}\sqrt{B-1}(B-A)^{2N-1}(\theta+B)^m}. \quad (3)$$

Here D_m, E_m are constants. Such a decomposition is possible, since $A \neq B$ in the half-strip Π_k. We next use the following lemma:

Lemma (I. V. Romanovskii, V. N. Sudakov). Let a finite number of measurable probability densities $p_m(x,y)$, $m = 1, 2, \ldots, M$, be given on the rectangle Q : $a \leq x \leq b$; $c \leq y \leq d$. Then for any prescribed $\alpha \in (0,1)$ there exists a measurable function $I(x,y)$, taking only the values 0 and 1, such that for almost all x (respectively y)

$$E^{(m)}(I(x,y) \mid x) = \alpha; \qquad E^{(m)}(I(x,y) \mid y) = \alpha$$

for $m = 1, 2, \ldots, M$; $E^{(m)}(\cdot \mid \cdot)$ denotes conditional expectation under the probability density $p_m(x,y)$.

Since α is different from 0 and 1, $I(x,y)$ is obviously nontrivial (not almost everywhere constant). Setting $\xi(x,y) = I(x,y) - \alpha$, we obtain a nontrivial measurable function with values $1-\alpha$ and $-\alpha$, such that

$$\int_a^b p_m(x,y)\xi(x,y)\,dy = 0; \qquad \int_c^d p_m(x,y)\xi(x,y)\,dx = 0 \quad (4)$$

for almost all x (respectively y). It is clear that the $p_m(x,y)$ may also be nonnormalized, but only such nonnegative functions that

$$\iint_Q p_m(x,y)\,dx\,dy < \infty$$

for $m = 1, 2, \ldots, M$.

Turning to the half-strip Π_k, we divide it into rectangles Q_s: $1 - 2^{-k} \le A < 1 - 2^{-k-1}$; $s \le B < s + 1$; $s = 1, 2, \ldots$. In each rectangle Q_s, according to the lemma, construct a function $\psi(A, B)$ with properties of type (4), where

$$p_m(A, B) = \frac{(AB)^{(n_1-3)/2}(B-A)^m}{\sqrt{1-A}\sqrt{B-1}(B-A)^{2N-1}}$$

is the unnormalized probability density in Q_s. Thus:

$$\int_{1-2^{-k}}^{1-2^{-k-1}} dA\, \psi(A, B) p_m(A, B) = \int_s^{s+1} dB\, \psi(A, B) p_m(A, B) = 0 \qquad (5)$$

for almost all B (respectively, A); $m = 1, 2, \ldots, N$.

From expression (3) we see that, for the function $\psi(A, B)$ defined in this way, the integral $J_k(\theta)$ will vanish for all $\theta > 0$, so that (2) will be true and, on the basis of what was said above, (1) is satisfied.

Thus the test $\phi(A, B) = \psi(A, B) + \alpha$ will be a nonrandomized measurable similar test of level α, which proves Theorem 1.

To prove Theorem 2, instead of N functions $p_m(A, B)$ we consider a finite number of functions $p_{mi}(A, B)$, where $i = 1, 2, \ldots, K$; for each i,

$$m \le N_i = \frac{n_{1i} + n_{2i} - 1}{2};$$

for given i and m, $p_{mi}(A, B)$ are constructed with the aid of n_{1i}, n_{2i}, and m in the same way as $p_m(A, B)$ with the aid of n_1, n_2, and m. Of course, the sufficient statistics $\bar{x}, \bar{y}, s_1, s_2$ will be taken for the given pair of samples, so that the test will depend on the number i of the pair of samples through them. We also note that, for the level $\alpha = 1/2$, a measurable nonrandomized similar test can be constructed by means of the theory of primes of a Gaussian field. Such a construction requires the use of the distribution law of the primes of a Gaussian field and is rather cumbersome.

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Note: Figure translations are in progress. See original paper for figures.