Mathematics

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Abstract

Full Text

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ON ONE CLASS OF MULTIDIMENSIONAL SINGULAR INTEGRAL EQUATIONS

Let $U(x,y,z)$ be a function harmonic in the ball $D: x^2+y^2+z^2<1$, with first derivatives continuous in the sense of Hölder in the closed ball $\overline{D}$. Denote the sphere $x^2+y^2+z^2=1$ by S.

For the function $U(x,y,z)$ the following integral representation holds:

$$U(x,y,z)=\int_0^x \varphi(t,y,z)\,dt+\omega(y,z), \tag{1}$$

where $\varphi(x,y,z)$ is a function harmonic in D, satisfying the boundary condition

$$\varphi(\xi,\eta,\zeta)=U_\xi(\xi,\eta,\zeta)=\psi(\xi,\eta,\zeta), \quad (\xi,\eta,\zeta)\in S, \tag{2}$$

and $\omega(y,z)$ is the general solution of the Poisson equation

$$\omega_{yy}+\omega_{zz}=-\varphi_x(0,y,z).$$

The function $\omega(y,z)$ is connected with the function $U(x,y,z)$ by the obvious relations

$$\omega(y,z)=U(0,y,z); \quad 2\operatorname{Re} U\left(0,\frac{y+iz}{2},\frac{y+iz}{2i}\right)+U(0,0,0)+\gamma(y,z)=0, \tag{3}$$

where $\gamma(y,z)$ is an arbitrary function harmonic in the cylinder $y^2+z^2<1$.

Using the well-known Poisson formula, the integral representation (1) may be put in the form

$$U(x,y,z)=\frac{1}{4\pi}\iint_S\left[\frac{(1-\xi^2-y^2-z^2)(x-\xi)}{\Delta R^{1/2}}+\frac{x+\xi}{R^{1/2}}-\operatorname{Arsh}\frac{x-\xi}{\Delta^{1/2}}\right]U_\xi\,dS+\gamma(y,z), \tag{4}$$

where

$$\Delta=(y-\eta)^2+(z-\zeta)^2, \qquad R=(x-\xi)^2+\Delta.$$

Formula (4) makes it possible to express the limiting values of the partial derivatives U_y and U_z on S from inside D through the limiting values of U_x in the form

$$U_y=-\frac{1}{2\pi}\iint_S\left[\frac{x\eta-\xi y}{\Delta R^{1/2}}+\frac{(x+\xi)(y-\eta)}{\Delta^2}R^{1/2}\right]U_\xi\,dS+\gamma_y, \tag{5}$$

$$U_z = -\frac{1}{2\pi} \iint_S \left[\frac{x\zeta - \xi z}{\Delta R^{1/2}} + \frac{(x+\xi)(z-\zeta)}{\Delta^2} R^{1/2} \right] U_\xi \, dS + \gamma_z. \tag{6}$$

The integrands in formulas (5) and (6) become infinite when $\xi = x, \eta = y, \zeta = z$ and when $\xi = -x, \eta = y, \zeta = z$, but under the assumptions made concerning the partial derivatives of the function $U(x, y, z)$, the integrals in the right-hand sides of these formulas exist in the sense of the principal value. In defining the integral in the principal-value sense

the points $(x, y, z) \in S$ and $(-x, y, z) \in S$ must be removed either by spheres of radius ε with centers at these points, or by means of a cylinder of radius ε whose axis passes through the indicated points.

Formulas (5) and (6) are a three-dimensional analogue of Hilbert' s formula, known from logarithmic potential theory, which relates the boundary values of the first partial derivatives of a function $U(x, y)$, harmonic in the disk $x^2+y^2 < 1$, on the circle $S : x^2 + y^2 = 1$:

$$U_y = \frac{1}{\pi} \int_S \frac{(x\eta - \xi y)U_\xi}{(x-\xi)^2 + (y-\eta)^2} \, dS + C = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{ctg} \frac{\theta - \theta_0}{2} U_\xi \, d\theta + C.$$

Let us now consider the singular integral equation

$$K\psi \equiv p\psi - \frac{q}{2\pi} \iint_S \left[\frac{x\eta - \xi y}{\Delta R^{1/2}} + \frac{(x+\xi)(y-\eta)}{\Delta^2} R^{1/2} \right] \psi \, dS -$$

$$- \frac{r}{2\pi} \iint_S \left[\frac{x\zeta - \xi z}{\Delta R^{1/2}} + \frac{(x+\xi)(z-\zeta)}{\Delta^2} R^{1/2} \right] \psi \, dS = f, \tag{7}$$

where p, q, r, and f are given functions on S satisfying a Hölder condition, while ψ is the unknown function, also satisfying a Hölder condition.

Assuming conditional or unconditional solvability of equation (7), denote by $U(x, y, z)$ the function harmonic in D that satisfies the boundary condition (2). The function $U(x, y, z)$ is given by formula (1).

Additionally imposing on the function $U(x, y, z)$ the condition

$$\operatorname{Re} U \left(0, \frac{y+iz}{2}, \frac{y+iz}{2i} \right) \equiv \text{const} \tag{8}$$

and using formulas (5) and (6), equation (7) can be written in the form

$$pU_x + qU_y + rU_z = f, \qquad (x, y, z) \in S. \tag{9}$$

Thus, the study of the singular integral equation (7) is reduced to the following oblique-derivative problem: find a function $U(x, y, z)$, harmonic in the ball D, with first partial derivatives continuous in the Hölder sense in $\overline{D}$, and satisfying the boundary condition (9) and condition (8).

It is sufficient to require that condition (8) be observed only for $y^2 + z^2 = 1$.

The oblique-derivative problem with boundary condition of the form (9) in the case of polynomial coefficients p, q, and r was investigated by us in papers ([1], [2]). Using the results of the indicated papers, we directly obtain the theory of singular integral equations of the form (7) with polynomial coefficients. Along with this, the question of regularization of the equation $K\psi + K_1\psi = f$, where K_1 is a completely continuous linear operator, is resolved in an obvious manner.

Let us consider two examples.

1. The coefficients p, q, and r are constants, with $p^2 + q^2 + r^2 \neq 0$. The solution of the oblique-derivative problem (9) in the case under consideration always exists and has the form

$$U(x, y, z) = U_0(x, y, z) + V(p_1 x + q_1 y + r_1 z,\ p_2 x + q_2 y + r_2 z), \tag{10}$$

where U_0 is a particular solution of the inhomogeneous problem, and V is the general solution of the corresponding homogeneous problem, with

$$\begin{Vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{Vmatrix}$$

a constant orthogonal matrix.

For $p \neq 0$, the function $U(x, y, z)$ determined by formula (10) can, by choosing V, always be made to satisfy condition (8), and, consequently, after such a choice the limiting values U_x on the sphere S will give the desired unique solution of equation (7). For $p = 0$ and $f \neq 0$, it will be impossible to satisfy condition (8) merely by choosing V, unless the function f is subjected to additional requirements. Thus, for $p = 0$ the homogeneous equation $K\psi = 0$ is always solvable (nonuniquely), whereas for solvability of the nonhomogeneous equation (7) the function f must be subjected to additional requirements following from (8).

2. $p = x - a,\ q = y,\ r = z$, where $a = \text{const}$. For $|a| > 1$, as is known (see (1)), the solution of problem (9) always exists and is given by the formula

$$U = U_0(x, y, z) + V\left(\frac{y}{x - a}, \frac{z}{x - a}\right), \tag{11}$$

where $U_0(x, y, z)$ is a particular solution of problem (9), and $V(\xi, \eta)$ is the general solution of the equation

$$(1 + \xi^2)V_{\xi\xi} + (1 + \eta^2)V_{\eta\eta} + 2\xi\eta V_{\xi\eta} + 2\xi V_{\xi} + 2\eta V_{\eta} = 0.$$

For $|a| < 1$, problem (9) is only conditionally solvable, and the corresponding homogeneous problem has no solution other than a constant.

From what has been said it follows that for $|a| > 1$, by choosing V, the function $U(x, y, z)$ represented by formula (11) can always be made to satisfy condition (8), so that equation (7) in the case under consideration is always solvable. If, however, $|a| < 1$, equation (7) is only conditionally solvable, i.e., for its solvability the function f must be subjected to additional requirements equivalent to the solvability conditions for problem (9) and to condition (8).

The results given above extend in an obvious way to the case of n independent variables.

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REFERENCES

[1] A. V. Bitsadze, DAN, **155**, No. 4 (1964). [2] A. V. Bitsadze, DAN, **157**, No. 6 (1964).

Note: Figure translations are in progress. See original paper for figures.