ON AN INEQUALITY FOR MODULI OF SMOOTHNESS OF A PERIODIC FUNCTION WITH BOUNDED DERIVATIVE

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Abstract

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ON AN INEQUALITY FOR MODULI OF SMOOTHNESS OF A PERIODIC FUNCTION WITH BOUNDED DERIVATIVE

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Let us consider, for some natural number p, the class $W_*^{(p)}$ of all periodic functions $f(x)$ of period 2π that have an absolutely continuous derivative of order $(p-1)$ and are such that $\left|f^{(p)}(x)\right| \leqslant 1$ almost everywhere. An important place in this class is occupied by the well-known Bernoulli functions

$$\varphi_p(x) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin[(2m+1)x - p\pi/2]}{(2m+1)^{p+1}}, \tag{1}$$

which arise in a number of extremal problems for periodic differentiable functions. The role of the functions (1) in such problems was first pointed out by S. N. Bernstein ([1]), who proved that if $f(x) \in W_*^{(p)}$ and

$$\int_0^{2\pi} f(x)\, dx = 0, \tag{2}$$

then

$$\max_x |f(x)| \leqslant \max_x \varphi_p(x). \tag{3}$$

The present note is devoted to the solution of one extremal problem in which the Bernoulli functions also occupy a central place.

Theorem 1. *For any natural number $k \leqslant p+1$ the equality*

$$\sup_{f \in W_*^{(p)}} \omega_k(f;t) = \omega_k(\varphi_p;t), \tag{4}$$

holds, where

$$\omega_k(f;t) = \max_{x,|h|\leqslant t} \left| \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x+\nu h) \right|$$

is the k-th modulus of smoothness of the function $f(x)$.

The proof of relation (4) is based on the known (see, for example, ([6]), p. 131) integral representation for functions $f(x) \in W_*^{(p)}$:

$$f(x) = \frac{1}{\pi} \int_0^{2\pi} f^{(p)}(t) \sum_{\nu=1}^{\infty} \frac{\cos[\nu(x-t) - p\pi/2]}{\nu^p} \, dt \tag{5}$$

and on the investigation of the sign of the differences of the kernel occurring under the integral in (5).

In the case $k = 1$, Theorem 1 follows easily from the work of A. F. Timan (see ([4]), Theorem 1).

For $k = p$, Theorem 1 and a theorem of D. A. Raikov ([2]) imply the validity of the following assertion, earlier stated by A. F. Timan without proof at his seminar on the theory of functions.

Theorem 2. *In order that a periodic function of period 2π belong to the class $W_*^{(p)}$, it is necessary and sufficient that, for every $t \geqslant 0$, it satisfy the condition*

$$\omega_p(f;t) \leqslant \omega_p(\varphi_p;t). \tag{6}$$

We note that Theorem 1 loses its force for $k > p+1$. This is seen from the following example.

Let $k = 3$ and $p = 1$. If we set

$$f^{(1)}(\pi + x) = \begin{cases} -1, & 0 \leq x \leq \frac{1}{4}\pi, \\ 1, & \frac{1}{4}\pi < x \leq \frac{3}{4}\pi, \\ -1, & \frac{3}{4}\pi < x \leq \frac{5}{4}\pi, \\ 1, & \frac{5}{4}\pi < x \leq \frac{7}{4}\pi, \\ -1, & \frac{7}{4}\pi < x \leq 2\pi, \end{cases}$$

then it is not hard to verify that, for $\frac{1}{3}\pi < t < \frac{2}{3}\pi$, $\omega_3(f;t) > \omega_3(\varphi_1;t)$.

It can be shown that the value of the right-hand side of (4), for all nonnegative $t \leq \pi$ and $k \leq p+1$, is determined by the equality

$$\omega_k(\varphi_p;t) = \frac{4}{\pi} \left| \sum_{\nu=0}^{\infty} (-1)^{\nu(p+k+1)} \frac{\{2\sin(2\nu+1)t/2\}^k}{(2\nu+1)^{p+1}} \right|. \tag{7}$$

Denote by $\widetilde{W}_*^{(p)}$ the class of all functions $f(x)$, periodic with period 2π, trigonometrically conjugate to functions from $W_*^{(p)}$. Along with Theorem 1, we also note the following proposition.

Theorem 3. *For any natural $k \le p$ the equality*

$$\sup_{f \in \widetilde{W}_*^{(p)}} \omega_k(f;t) = \omega_k(\widetilde{\varphi}_p;t),$$

holds, where $\widetilde{\varphi}_p(x)$ is the function trigonometrically conjugate to $\varphi_p(x)$.

If $0 \le t \le \pi$, then

$$\omega_k(\widetilde{\varphi}_p;t) = \frac{4}{\pi}\left|\sum_{\nu=0}^{\infty}(-1)^{\nu(p+k)}\frac{\{\sin(2\nu+1)t/2\}^k}{(2\nu+1)^{p+1}}\right|. \tag{8}$$

From Theorem 3, for $p = 1$, $k = 1$, there immediately follows a corollary sharpening a known theorem of I. I. Privalov (see, for example, ([5]), Ch. VII, p. 59), namely, that if

$$|f(x) - f(x+h)| \le h,$$

then

$$|\tilde{f}(x) - \tilde{f}(x+h)| = O\left(h \ln \frac{1}{h}\right).$$

Corollary 1. *If a periodic function $f(x)$ of period 2π satisfies the Lipschitz condition*

$$|f(x) - f(x+h)| \le h,$$

then for the function $\tilde{f}(x)$, trigonometrically conjugate to $f(x)$, for any nonnegative $t \le \pi$ the inequality

$$\omega(\tilde{f};t) \le \frac{2}{\pi} t \ln \operatorname{ctg}\frac{t}{4} + \frac{4}{\pi}\int_0^{t/2} \frac{x}{\sin x}\,dx, \tag{9}$$

holds, and it is sharp.

Let $W_*^{(p)}(L)$ be the class of functions $f(x)$, periodic with period 2π, having an absolutely continuous derivative of order $(p-1)$ and such that

$$\int_0^{2\pi} |f^{(p)}(t)|\, dt \le 1.$$

Let, further, $\widetilde{W}_*^{(p)}(L)$ be the class of all functions $f(x)$ of period 2π trigonometric-conjugate to functions from $W_*^{(p)}(L)$.

Then the following theorems hold:

Theorem 4. For any natural $k \le p+1$ the equality

$$\sup_{f \in W_*^{(p)}(L)} \omega_k(f;t)_L = \omega_k(\varphi_p;t), \tag{10}$$

holds, where $\omega_k(f;t)_L$ is the integral modulus of smoothness:

$$\omega_k(f;t)_L = \sup_{|h|\le t} \int_0^{2\pi} \left| \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x+\nu h) \right| dx.$$

Theorem 5. For natural $k \le p$ the equality

$$\sup_{f \in \widetilde{W}_*^{(p)}(L)} \omega_k(f;t)_L = \omega_k(\varphi_p;t) \tag{11}$$

holds.

The validity of relations (10) and (11) follows from Theorems 1 and 3 and from an equality of S. M. Nikol' skii (see [3], Theorem 3):

$$\sup_{f \in W_*^{(p)}(L)} \int_0^{2\pi} \left| \int_0^{2\pi} f^{(p)}(t) \sum_{\nu=0}^{\infty} \frac{\cos[\nu(x-t) - p\pi/2]}{\nu^p}\, dt \right| dx =$$

$$= \frac{1}{2} \max_t \int_0^{2\pi} \left| \sum_{\nu=0}^{\infty} \frac{\cos[\nu(x-t) - p\pi/2] - \cos[\nu x + p\pi/2]}{\nu^p} \right| dx.$$

From Theorem 5, for $p = 1,\ k = 1$, the following corollary follows, refining a theorem analogous to a theorem of I. I. Privalov.

Corollary 2. If a periodic function $f(x)$ of period 2π satisfies the condition

$$\int_0^{2\pi} |f(x) - f(x+t)|\, dx \le t,$$

then for the function $\tilde{f}(x)$, trigonometric-conjugate to $f(x)$, for any nonnegative $t \le \pi$ the inequality

$$\omega(\tilde{f};t)_L \le \frac{2}{\pi}\, t \ln \operatorname{ctg} \frac{t}{4} + \frac{4}{\pi} \int_0^{t/2} \frac{x}{\sin x}\, dx, \tag{12}$$

holds, and it is sharp.

In conclusion, I consider it my duty to express deep gratitude to Prof. A. F. Timan for posing the problem and for his constant attention.

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Note: Figure translations are in progress. See original paper for figures.