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Abstract

Full Text

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ON CONDITIONS FOR SOLVABILITY OF THE ABSTRACT CAUCHY PROBLEM

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Let us consider* the differential equation

$$\frac{dx(t)}{dt} = Ax(t) \qquad (0 \le t < T;\ T \le \infty) \tag{1}$$

with a linear operator A in a Banach space, and pose the question: for which vectors x_0 does there exist a solution of equation (1) satisfying the initial condition

$$x(0) = x_0. \tag{2}$$

In order to answer this question, we shall generalize the classical Laplace transform in such a way that the generalized (or, as we shall call it, local) Laplace transform is applicable on a finite time interval $[0, T)$, as well as on the infinite interval $[0, \infty)$ without restrictions on growth.

Let $f(\lambda)$ be a locally integrable** vector-valued function defined for all sufficiently large $\lambda > 0$, say for $\lambda \ge \lambda_f$. We shall say that the function $f(\lambda)$ belongs to the class Λ_T $(0 < T \le \infty)$ if there exists a vector-valued function $\tilde{f}(\tau, t)$ $(0 \le \tau \le t,\ 0 < t < T)$, integrable** with respect to τ, such that for every $t \in (0, T)$ the representation

$$f(\lambda) = \int_0^t \tilde{f}(\tau, t) e^{-\lambda\tau}\, d\tau + \varepsilon(\lambda, t)$$

holds, with a remainder term $\varepsilon(\lambda, t)$ satisfying the condition

$$\overline{\lim_{\lambda \to +\infty}} \lambda^{-1} \ln \|\varepsilon(\lambda, t)\| \le -t.$$

We shall call the function $\tilde{f}(\tau, t)$ the **local Laplace original** for $f(\lambda)$, and $f(\lambda)$ the **local Laplace transform (image)** for $\tilde{f}(\tau, t)$.

Lemma 1. *For a given function $f(\lambda) \in \Lambda_T$, the local Laplace original $\tilde{f}(\tau, t)$ is uniquely determined*** and does not depend on the parameter t.*

Therefore, in what follows one may write $\tilde{f}(\tau)$ instead of $\tilde{f}(\tau, t)$. The proof of this lemma is based on the fact (see, for example, ([2]), pp. 412–413) that if a scalar function $\psi(t)$ $(0 \le t \le a,\ a < \infty)$, summable on $[0, a]$, is such that

$$\overline{\lim_{\lambda \to +\infty}} \lambda^{-1} \ln \left| \int_0^a \psi(t) e^{\lambda t}\, dt \right| \le b,$$

where $0 \le b < a$, then $\psi(t) = 0$ almost everywhere on (b, a).

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* For definiteness we consider the so-called ACP (([1]), pp. 631-632), but the method we use is general.

** In the strong sense (see, for example, ([1]), pp. 92–103).

*** This cannot be said of the local Laplace transform.

The following proposition describes an analytic procedure for inverting the local Laplace transform*.

Lemma 2. Let $f(\lambda) \in \Lambda_T$, and let $\tilde{f}(\tau)$ be the local Laplace original for $f(\lambda)$. Put

$$F(t) = \frac{1}{2\pi i} \int_a^\infty \frac{f(\lambda) e^{\lambda t}}{\lambda}\, d\lambda \qquad (a \ge \lambda_f,\ a > 0;\ \operatorname{Re} t < 0).$$

The function $F(t)$ is analytically continued to the domain

$$\Pi_T = \{t \mid \operatorname{Re} t < T,\ t \notin [0, T)\}$$

and

$$\lim_{\sigma \downarrow 0} [F(t + i\sigma) - F(t - i\sigma)] = \int_0^t \tilde{f}(\tau)\, d\tau \qquad (0 < t < T). \tag{3}$$

Another method of inverting the local Laplace transform, not requiring complex variables, follows from a formula of Phragmén ([3]); ([4]), pp. 21-23), but we shall not dwell on this.

Denote by $\Lambda_T^{(1)}$ the subclass of the class Λ_T singled out by the requirement of smoothness (in the strong sense) of the original $\tilde{f}(\tau)$ $(0 \le \lambda < T)$. In this subclass the inversion formula is simplified. Namely, instead of $F(t)$ one should consider the function

$$\Phi(t) = \frac{1}{2\pi i} \int_{\lambda_f}^\infty f(\lambda) e^{\lambda t}\, d\lambda,$$

and instead of (3) one may write

$$\Phi(t+i0)-\Phi(t-i0)=\tilde{f}(t) \qquad (0<t<T). \tag{4}$$

We now turn to the abstract Cauchy problem (1)–(2). Suppose that there exists a real $\lambda=\lambda_A$ such that the values $\lambda\geq\lambda_A$ do not belong to the spectrum of the operator A.

Theorem 1. If the resolvent R_λ of the operator A satisfies the condition**

$$h_A \equiv \overline{\lim_{\lambda\to+\infty}}\, \lambda^{-1}\ln\|R_\lambda\|<\infty, \tag{5}$$

then $R_\lambda x_0\in\Lambda^{(2)}_{T-h_A}$ for all those vectors x_0 for which the problem (1)–(2) $(T>h_A)$ has a smooth solution.

Proof. Apply the operator R_λ to both sides of equation (1):

$$\frac{dR_\lambda x(t)}{dt}=x(t)+\lambda R_\lambda x(t) \qquad (0\leq t<T).$$

Hence

$$R_\lambda x_0=-\int_0^t x(\tau)e^{-\lambda\tau}\,d\tau+e^{-\lambda t}R_\lambda x(t) \qquad (0\leq t<T). \tag{6}$$

In view of (5), $R_\lambda x_0\in\Lambda^{(1)}_{T-h_A}$, and, analogously to the classical situation, the local Laplace transform for $R_\lambda x_0$ is equal to $-x(\tau)$.

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* We note that the local Laplace original coincides with the classical one if the latter exists.

** This condition ensures, according to the results of ([5],[6]), the uniqueness of the solution of problem (1)–(2).

Corollary. Let condition (5) be satisfied. Put

$$K_t=\frac{1}{2\pi i}\int_{\lambda_A}^{\infty}R_\lambda e^{\lambda t}\,d\lambda \qquad (\operatorname{Re}t<-h_A).$$

If $x(t)$ is a smooth solution of problem (1)–(2), then the vector-function $K_t x_0$ is analytically continued into the domain Π_{T-h_A}, and there

$$K_t x_0\Big|_{t-i0}^{t+i0}=x(t) \qquad (0<t<T-h_A).$$

This proposition substantially supplements the uniqueness theorems obtained by the author in ([5,6]).

Theorem 1 can be inverted in the following sense.

Theorem 2. If the vector x_0 is such that $R_\lambda x_0 \in \Lambda_T^{(1)}$, then problem (1)–(2) has a smooth solution.

Here nothing is assumed about the behavior of $\|R_\lambda\|$ as $\lambda \to +\infty$.

Proof. Denote the local Laplace original of the function $R_\lambda x_0$ by $-x(\tau)$ $(0 \leq \tau < T)$:

$$R_\lambda x_0 = -\int_0^t x(\tau)e^{-\lambda\tau}\,d\tau + e^{-\lambda t}\eta(\lambda,t) \qquad (0 \leq t < T). \tag{7}$$

Here

$$\overline{\lim_{\lambda\to+\infty}} \lambda^{-1}\ln\|\eta(\lambda,t)\| \leq 0. \tag{8}$$

We shall show that the function $x(t)$ satisfies equation (6) (i.e., that $\eta(\lambda,t) = R_\lambda x(t)$).

Apply to both sides of relation (7) the operator R_μ $(\mu \geq \lambda_A)$ and use Hilbert' s identity

$$R_\mu R_\lambda = \frac{R_\lambda - R_\mu}{\lambda - \mu}.$$

Putting $\lambda - \mu = \omega$, we obtain:

$$R_\lambda x_0 - R_\mu x_0 = -\omega\int_0^t e^{-\lambda\tau}R_\mu x(\tau)\,d\tau + \omega e^{-\lambda t}R_\mu\eta(\lambda,t).$$

Hence, by virtue of (7),

$$\omega\int_0^t e^{-\lambda\tau}R_\mu x(\tau)\,d\tau - \int_0^t x(\tau)e^{-\lambda\tau}\,d\tau = R_\mu x_0 + [\omega R_\mu\eta(\lambda,t) - \eta(\lambda,t)]e^{-\lambda t}. \tag{9}$$

But

$$\int_0^t x(\tau)e^{-\lambda\tau}\,d\tau = e^{-\omega t}\int_0^t x(\tau)e^{-\mu\tau}\,d\tau + \omega\int_0^t e^{-\omega\tau}\,d\tau\int_0^\tau x(\sigma)e^{-\mu\sigma}\,d\sigma. \tag{10}$$

Putting

$$\rho(\mu,\tau) = e^{-\mu\tau} R_\mu x(\tau) - \int_0^\tau x(\sigma) e^{-\mu\sigma}\, d\sigma,$$

we obtain from (9) and (10)

$$\omega \int_0^t e^{-\omega\tau} \rho(\mu,\tau)\, d\tau = R_\mu x_0 + e^{-\omega t} \int_0^t x(\tau) e^{-\mu\tau}\, d\tau + [\omega R_\mu \eta(\lambda,t) - \eta(\lambda,t)] e^{-\lambda t},$$

whence

$$\int_0^t e^{\omega\tau} \left[\rho(\mu, t-\tau) - R_\mu x_0\right] d\tau =$$

$$= \frac{1}{\omega} R_\mu x_0 + \frac{1}{\omega} \int_0^t x(\tau) e^{-\mu\tau}\, d\tau + \left[R_\mu \eta(\mu+\omega,t) - \frac{1}{\omega}\eta(\mu+\omega,t)\right] e^{-\mu t}.$$

Fixing μ and letting ω tend to $+\infty$, we obtain, by virtue of (8),

$$\varlimsup_{\omega\to+\infty} \frac{\ln\left\|\int_0^t e^{\omega\tau}[\rho(\mu,t-\tau) - R_\mu x_0]\, d\tau\right\|}{\omega} \le 0.$$

It follows from this that

$$\rho(\mu, t-\tau) = R_\mu x_0 \qquad (0 \le \tau \le t,\ \mu \ge \lambda_A),$$

i.e., that $x(t)$ does indeed satisfy equation (6). Differentiating (6) with respect to t, we obtain

$$R_\lambda \left[\frac{dx(t)}{dt} - \lambda x(t)\right] = x(t),$$

whence it follows that $x(t) \in D_A$ and equation (1) holds. Setting $t = 0$ in (6), we arrive at equality (2). Thus the theorem is completely proved.

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REFERENCES

1. E. Hille, R. Phillips, *Functional Analysis and Semi-Groups*, IL, 1962.
2. E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Moscow-Leningrad, 1948.
3. E. H. Phragmén, *Acta Math.*, **28**, 331 (1904).
4. J. Mikusiński, *Operational Calculus*, IL, 1956.
5. Yu. I. Lyubich, DAN, **130**, No. 5, 969 (1960).
6. Yu. I. Lyubich, UMN, **16**, No. 5, 111 (1961).

Note: Figure translations are in progress. See original paper for figures.