Reports of the Academy of Sciences of the USSR

1964

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1964. Volume 158, No. 5

MATHEMATICS

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ON THE SPECTRUM OF NORMAL OPERATORS AND THEIR PERTURBATIONS

(Presented by Academician L. S. Pontryagin on 29 IV 1964)

The present paper generalizes the results of the author' s preceding note (^{1}) and some lemmas of I. M. Glazman (^{2}). We consider normal operators in a Hilbert space.

Theorem 1. Let A and B be normal operators and let the spectrum $\sigma(B)$ of the bounded operator B have the property

$$\sigma(B) \subset \{z : \ |z-a| \leqslant r\}.$$

a) If there is a manifold $F \subset D_A$ of dimension n, $1 \leqslant n \leqslant \infty$, such that

$$\|(A-B)f\| \leqslant \alpha\|f\| \quad \text{for all } f \in F,$$

then $\{z : \ |z-a| \leqslant r+\alpha\}$ contains at least n points of the spectrum $\sigma(A)$.

b) If in $\{z : \ |z-a| \leqslant r+\alpha\}$ there are n points of the spectrum $\sigma(A)$, $1 \leqslant n \leqslant \infty$, then there is a manifold F of dimension n such that

$$\|(A-B)f\| \leqslant (2r+\alpha)\|f\| \quad \text{for all } f \in F.$$

c) If there is a subspace G with defect number n, $1 \leqslant n < \infty$, such that

$$\|(A-B)g\| > \alpha\|g\| \quad \text{for all } g \in G, \quad \alpha > r,$$

then the number of points of the spectrum $\sigma(A)$ in $\{z : \ |z-a| \leqslant \alpha - r\}$ is less than or equal to n.

d) If the number of points of the spectrum $\sigma(A)$ in $\{z : \ |z-a| \leqslant \alpha+r\}$ is less than or equal to $n < \infty$, then there is a subspace G with defect number not greater than n, for which

$$\|(A-B)g\| > \alpha\|g\| \quad \text{for all } g \in G.$$

The theorem remains true if in a), b), c), d) we replace all $\leqslant$ by $<$ and conversely (with the exception of the inequalities for n).

In the case $r = 0$, i.e. $B = aE$, the theorem gives necessary and sufficient conditions (^{2}).

Similar theorems hold not only for bounded operators B. We shall give two special lemmas.

Lemma 1. Let A and B be normal operators,

$$\sigma(B) \subset \{z:\ |z-a| \geqslant r\}.$$

If there is $F \in D_A \cap D_B$, $\|f\| = 1$, $\|Af - Bf\| \leqslant \alpha < r$, then

$$\sigma(A) \cap \{z:\ |z-a| \geqslant r-\alpha\} \neq \emptyset.$$

Lemma 2. Let A and B be normal operators,

$$\sigma(B) \subset \{z:\ \operatorname{Re} e^{i\varphi}(z-a) \geqslant 0\} = K, \qquad 0 \leqslant \varphi < 2\pi.$$

If there is $f \in D_A \cap D_B$, $f \neq 0$, such that $Af = Bf$, then $\sigma(A) \cap K \neq \emptyset$.

The proofs of Theorem 1 and of these lemmas are analogous to the proof in (^{1}). The first part of Theorem 1 in (^{1}) is a consequence of Theorem 1a), since the multiplication operator is a normal operator.

Theorem 2. Let A be a normal operator and, for $f_0 \in D_A$, $f_0 \neq 0$,

$$Af_0 = f_1.$$

Let K_c be the circle with center

$$a = \frac{a_1}{a_0} + \frac{x\delta - 1}{2ca_0}$$

and radius $r =$

$$= \frac{c\delta + 1}{2|c|a_0}, \qquad \text{where } a_0 = (f_0, f_0), \quad a_1 = (f_1, f_0), \quad a_2 = (f_1, f_1), \quad \delta = a_2a_0 - a_1\overline{a_1},$$

$c \neq 0$ is an arbitrary complex number. Then $K_c \cap \sigma(A) \neq \emptyset$.

Corollary. For fixed f_0, with $c = \delta^{-1/2}$, we obtain the minimal radius, namely, $\hat{a} = a_1/a_0,\ \hat{r}^2 = \delta/a_0^2$.

These circles are known as Krylov-Bogolyubov circles.

Theorem 3. Let A be a normal operator in $L_2(D)$, $D \subset R^n$ a measurable set of positive measure, and let $q(x)$ be a measurable function on D. Let Q be the closed convex hull of the spectrum of the function $q(x)$, and let K be the smallest (closed) circle containing Q. Let

$$\widetilde{A}f = Af + qf$$

for $f \in D_{\widetilde{A}} = D_A$ be the perturbed operator, and let $\tilde{\sigma}(\widetilde{A})$ be the subset of all points of $\sigma(\widetilde{A})$ that do not belong to the pure residual spectrum. Then

$$\tilde{\sigma}(\widetilde{A}) \subset M = \{z : \ z = z_1 + z_2,\ z_1 \in \sigma(A),\ z_2 \in K\}.$$

Let M' be the set of all z for which one can find a circle K_z, $\{z - Q\} \subset K_z$, $K_z \cap \sigma(A) = \emptyset$; then $M' \subset M$ and even $\tilde{\sigma}(\widetilde{A}) \subset M'$.

Corollary 1. Suppose there is a sequence $\{z_n\}$, $z_n \in \sigma(A)$, such that

$$\lim_{n\to\infty} d(z_n) = \infty, \qquad \text{where } d(z_n) = \inf_{\substack{z\in\sigma(A)\\ z\neq z_n}} |z_n - z|.$$

Then the components of the set M' corresponding to the points z_n asymptotically have the form $\{z_n + Q\}$. These components, generally speaking, are not convex.

Corollary 2. Let A be a self-adjoint regular differential operator (with discrete spectrum) and $q(x)$ a real, bounded, continuous function, $|q(x)| \le b$, and let

$$\widetilde{A}f = Af + iqf$$

for $f \in D_{\widetilde{A}} = D_A$ be the perturbed operator. Then M' contains $\sigma(\widetilde{A})$ and is described explicitly. For $z_n \in \sigma(A)$ define $K_n = K_n^l \cup K_n^r$. If $a_n = |z_n - z_{n-1}| \le 2b$, then $K_n^l = K \cap \{z : \operatorname{Re} z \le 0\}$; if $a_n > 2b$, then K_n^l is the closed domain bounded by the imaginary axis and the right branch of the hyperbola

$$\left(x + \frac{a_n}{2}\right)^2 - y^2 = \frac{1}{4}(a_n^2 - 4b^2).$$

Analogously, K_n^r is either $K \cap \{z : \operatorname{Re} z \ge 0\}$, or the domain bounded by the imaginary axis and the left branch of the hyperbola

$$\left(x - \frac{a_{n+1}}{2}\right)^2 - y^2 = \frac{1}{4}(a_{n+1}^2 - 4b^2).$$

Then

$$M' = \bigcup_n \{z_n + K_n\}.$$

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Received
27 IV 1964

References

1. K. P. Hadeler, DAN, **157**, No. 2 (1964).

2. I. M. Glazman, *Direct methods of qualitative spectral analysis of singular differential operators*, Moscow, 1964.

Note: Figure translations are in progress. See original paper for figures.

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