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Abstract

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APPLICATION OF THE METHOD OF STATIONARY PHASE TO THE SOLUTION OF KINETIC EQUATIONS

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§ 1. In the theory of multiple scattering ([2–4]), kinetic equations of the form are considered:

$$\frac{\partial u(\mathbf{n},t)}{\partial t} = \int w(\mathbf{n}\mathbf{n}')[u(\mathbf{n}',t) - u(\mathbf{n},t)]\,d\Omega(\mathbf{n}'); \tag{1}$$

here $\int w(\mathbf{n}\mathbf{n}')\,d\Omega(\mathbf{n}') = 1$; the function $w(\mathbf{n}\mathbf{n}')$ is proportional to the scattering cross section; t is the path traversed by the particle; the scattering process in (1) is assumed to proceed without energy loss. It is clear that the solution of equation (1) with the initial condition

$$u(\mathbf{n},0) = \delta(\mathbf{n} - \mathbf{n}_0) \tag{2}$$

can be represented by the series ([4]):

$$u(\mathbf{n},t) = \sum \frac{2l+1}{4\pi} P_l(\mathbf{n}\mathbf{n}_0)a_l(t), \tag{3}$$

where

$$a_l(t) = \exp[t(w_l - 1)], \qquad w_l = 2\pi \int_{-1}^{1} w(t)P_l(t)\,dt.$$

Formally, (3) gives the general solution of the problem of multiple scattering. In practically interesting cases, however, in (3) it is necessary to take into account a very large number of terms. This is inconvenient* and compels one to seek other methods for computing the function (3).

Molière ([3]), in the case of scattering of electrons in a screened Coulomb potential, gave an approximate expression for the sum of the series (3), suitable in the region of small deflection angles. In some cases it is convenient to use an expansion of $u(\mathbf{n}, t)$ in the number of collisions experienced, or to take scattering through large angles into account as scattering of small multiplicity with corrections due to multiple scattering ([5]).

In the present work we give a new method for computing the solution of equation (1) and of more complicated equations corresponding to allowance for energy losses of particles during scattering. The principal advantage of the proposed method is that it (in the variant of § 2) makes it possible to obtain the distribution function $u(\mathbf{n}, t)$ over the entire interval of angles. Our method consists in the following. We transform formula (3) to the form ([6])

$$u(\mathbf{n}, t) = \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} \frac{\rho\, d\rho}{4\pi i} \left\{ \frac{P_{-1/2+\rho}(\pi-\theta)}{\cos\pi\rho}\, a_{-1/2+\rho}(t) \right\}; \tag{4}$$

* Especially when scattering proceeds with loss of energy, since in this case the computation of the coefficients $a_l(t)$ becomes considerably more complicated; for deflections through large angles, if such deflections are improbable, it may happen that the unavoidable errors in computing $a_l(t)$, in view of large cancellations, lead to an error in the sum of the series (3) greater than the sum itself.

here $\cos\theta = \mathbf{n}\mathbf{n}_0$. The function $P_{-1/2+i\alpha}(\cos\varphi)$ for $\alpha > 0$ is positive and increases with increasing α like $\sim e^{\alpha\varphi}$; therefore the expression in braces in (4) has a saddle point $\rho = \rho_0(\theta, t)$ in the upper imaginary half-plane. As an approximation to $u(\mathbf{n}, t)$ we take the integral (4), taken over a contour passing through the point $\rho_0(\theta, t)$, on which the integrand preserves a constant phase*. Thus, the proposed method is, generally speaking, numerical. We are not in a position to give any sufficiently accurate justification of our method and shall confine ourselves only to presenting the considerations that led to it, as well as the results of numerical calculation confirming it; moreover, it is precisely the latter that we regard as our main argument.

§ 2. We begin with the simplest one-dimensional equation of the form

$$\frac{\partial u(x,t)}{\partial t} = \int w(x-x')\,[u(x',t) - u(x,t)]\, dx'. \tag{5}$$

Instead of (3) we shall have

$$u = \frac{1}{2\pi} \int \exp\{ipx + t[\widetilde{w}(p) - 1]\}\, dp, \tag{6}$$

where

$$\widetilde{w}(p) = \int w(x) \exp(ipx)\, dx.$$

Expanding

$$\widetilde{w}(p) = 1 - \frac{\alpha^2 p^2}{2} + \cdots, \qquad \alpha^2 = \int x^2 w(x)\, dx$$

(we consider the function $w(x)$ finite and even), we find the asymptotics of u as $t \to \infty$:

$$u \sim (4\pi\alpha t)^{-1/2} \exp\left[-\frac{x^2}{4\alpha t}\right]. \tag{7}$$

This is a well-known device, and our device of replacing the contour of integration in (6) by a contour of stationary phase is, evidently, its direct generalization. The corresponding result of its application, as it seems to us, may be, in comparison with (7), only an enlargement of the region $0 < x < x_0(t)$ in which the estimate obtained is satisfactory. (Namely, for (7) $x_0(t) \sim t^{3/4}$; for our method, when $w(x) = 0$, $|x| > 1$ and $w(x) = 1$, $|x| < 1$, we obtained $x_0(t) \sim t$.) The quantity t, which here plays the role of the mean number of collisions, we regard as fixed and, generally speaking, large, for the function u, if $w(x)$ is nonanalytic, can only for large t be well approximated by the analytic function given by our method.

Let now the random process described by equation (5) occur not on a straight line, but on a circle. In this case the functions $u(x)$ and $w(x)$ should be considered periodic with period 2π; the expansion (6) is replaced by

$$u(x,t) = \frac{1}{2\pi} \sum \exp\{imx + t[\widetilde{w}(m) - 1]\},$$

where

$$\widetilde{w}(m) = \int_{-\pi}^{\pi} w(x) \exp(imx)\, dx.$$

Similarly to (3) and (4), the representation of u obtained can be transformed to the form

$$u(x,t) = \frac{1}{2\pi} \int_L \frac{\exp\{imx + t[\widetilde{w}(m) - 1]\}}{[\exp(2\pi i m) - 1]}\, dm, \tag{8}$$

where L is a contour composed of the straight lines $\mathrm{Im}(m) = \pm\varepsilon$, traversed counterclockwise.

We shall separately consider the integrals u_+ and u_- over the upper and lower parts of this contour; we shall assume that $0 < x < \pi$. For $\mathrm{Im}(m) = \varepsilon > 0$

* Or an estimate of this integral by the contribution from the saddle point.

$$|\exp(2\pi i u)| < 1,$$

therefore

$$u_+(x,t) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\int_{-\infty}^{+\infty} dm\, \exp\left[imx + 2\pi ikm + t(\widetilde{w}(m) - 1)\right] = \sum_{k=0}^{\infty} u_{+k}.$$

Similarly,

$$u_-(x,t) = \sum_{0}^{\infty} u_{-k}(x,t),$$

where

$$u_{-k} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dm\, \exp\left[imx - 2\pi im(k+1) + t(\widetilde{w}(m) - 1)\right].$$

Since $w(-x) = w(x)$ and $\widetilde{w}(x) \geqslant 0$, the function $w(m)$ is positive and increases monotonically with increasing $|\,\mathrm{Im}(m)|$ for $\mathrm{Re}\, m = 0$. It follows that the integrand of each of the integrals $u_{\pm k}$ has a saddle point on the imaginary axis, and the quantity $u_{\pm k}$ can be estimated by this saddle point. Note that increasing the index k by one corresponds to increasing the scattering angle by 2π. Thus, in order that in the expansion $u = \sum(u_{+k} + u_{-k})$ it be possible to restrict oneself to the terms with $k = 0$ (so that the application of the proposed method does not become too complicated), it is necessary that the function u be small for large angles $x \sim \pi$.

We now obtain a representation analogous to (8), when the random process takes place on the surface of a sphere. For this purpose we substitute into (4) the integral representation

$$P_{-1/2+\rho}[\cos(\pi - \theta)] = \frac{1}{\pi\sqrt{2}}\int_{\theta-\pi}^{\pi-\theta} \frac{e^{i\rho\alpha}\, d\alpha}{\sqrt{\cos\alpha + \cos\theta}},$$

in which we split the integral into two parts: from 0 to $\pi - \theta$ and from $\theta - \pi$ to 0. Thus,

$$u(\mathbf{n},t) = \frac{1}{4\pi^2 i\sqrt{2}}\int_0^{\pi-\theta} \frac{d\alpha}{\sqrt{\cos\alpha + \cos\theta}}\int_L \frac{\rho\, d\rho}{\cos\pi\rho}\, e^{i\rho\alpha} a_{\rho-1/2};$$

here the contour L is the same as in (8). Expanding the denominator as before, we obtain

$$u = \frac{1}{2\pi^2\sqrt{2}}\int_0^{\pi-\theta} \frac{d\alpha}{\sqrt{\cos\theta + \cos\alpha}}\left[-F(\pi - \alpha) + F(\pi + \alpha) - F(3\pi - \alpha)\ldots\right]; \quad (9)$$

here

$$F(z) = i \int_{-\infty}^{+\infty} \rho \, d\rho \, e^{i\rho z} a_{\rho-1/2}$$

since $F(0) = 0$, the expression in square brackets vanishes at $\alpha = \pi$, so that the integral defining u converges also for $\theta = 0$. The function $a_{\rho-1/2}$ is positive and increases monotonically with increasing $\operatorname{Im} \rho$ on the imaginary axis of ρ; therefore, similarly to $u_{\pm k}$, the function $F(z)$ can be estimated by the contribution from the saddle point. After this, the computation of u reduces to the calculation of a single integral with respect to α. As before, the condition for practical applicability of the method under consideration is the rapid decrease of the function $F(z)$ with increasing value of its argument (which ensures the smallness of u in the region of large angles). Taking $a_{\rho-1/2} = e^{-\tau^2\rho^2}$, it is not difficult to see that the more terms must be retained in (3), the fewer of them play a role in (9); such a relation

between expansions (3) and (9) apparently remains valid also for a more general choice of the function $a_{\rho-1/2}$.

The method indicated at the beginning of the article for calculating the function u is evidently a coarsened and simplified variant of the method given here. Its advantage is the absence of a superfluous integration with respect to a; its drawback is that it becomes unsuitable at small angles (since $P_{-1/2+\rho}(1) = \infty$). The remaining conditions for its applicability are the same: the number of collisions essential for scattering through the angle θ_0 of interest to us must be so large that the function u for $\theta = \theta_0$ can be well approximated by an analytic function; the distribution function u must be small in the region $\theta \sim \pi$.

§ 3. In connection with the problem of finding the angular and energy distribution of μ-mesons deep underground [1], we obtained by our method a solution of the kinetic equation

$$\frac{df(\mathbf{n},t)}{dt} = \frac{Q_1}{2\pi} \int_{yt^2<1-\mathbf{nn}'<xt^2} \frac{d\Omega(\mathbf{n}')}{(1-\mathbf{nn}')^2} \left[f(\mathbf{n}',t) - f(\mathbf{n},t)\right] + \frac{Q_2}{2\pi} \int_{xt^2<1-\mathbf{nn}'<t} d\Omega(\mathbf{n}') \left[f(t-1+\mathbf{nn}',\mathbf{n}') - f(t,\mathbf{n})\right] \frac{}{(1} \tag{10}$$

here $Q_1 = 2.3 \cdot 10^{-4}$, $Q_2 = 2.0 \cdot 10^{-5}$, $x = 1.2 \cdot 10^{-3}$, $y = 1.5 \cdot 10^{-11}$; this time t is the wavelength of the μ-meson. This equation describes multiple scattering of particles of zero rest mass in a Coulomb potential, taking into account the loss of energy to the recoil of target particles. For $t = 1.33$ and $\cos\theta$ equal to 0.9, 0.8, etc. down to -0.3, the calculation gave (in the numerator—the decimal logarithm with the opposite sign of the solution of equation (10), found by our method, with initial condition (2); in the denominator—the same for the solution in the single-scattering approximation):

$$f_1 = \max\left\{0, \frac{3.12 \cdot 10^{-6}}{(1-\cos\theta)^2}(t + \cos\theta - 1)\right\}$$

$$\frac{1.54}{3.41};\quad \frac{2.37}{4.06};\quad \frac{3.01}{4.45};\quad \frac{3.55}{4.74};\quad \frac{4.07}{4.98};\quad \frac{4.54}{5.20};\quad \frac{4.97}{5.40};\quad \frac{5.40}{5.59};\quad \frac{5.70}{5.78};\quad \frac{5.95}{5.99};\quad \frac{6.30}{6.23};\quad \frac{6.70}{6.55};\quad \frac{7.75}{7.20}.$$

The ratio of the "exact" and "single-scattering" solutions turns out to be quite plausible (except that near angles close to the limiting angle of single scattering, the exact value turns out to be less than the single-scattering one). In favor of our method is also the fact that it correctly "senses" the limiting angle of single scattering. We note that in our calculation the integral along the curve of stationary phase could always be estimated with sufficient accuracy by the formula

$$\int e^{\psi(z)}\,dz \sim \sqrt{\frac{2\pi}{\psi''(z_0)}}\,e^{\psi(z_0)};$$

the error was $\sim 10\%$ in the direction of underestimation. This observation makes it possible to simplify the calculation still further.

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Note: Figure translations are in progress. See original paper for figures.