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## Abstract

## Full Text

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# ON A UNIQUENESS THEOREM

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1°. As usual, by $A$ we denote the class of functions $w(\xi)$ regular in $|\xi| < 1$ for which the integral

$$\int_0^{2\pi} \lg^+ |w(re^{i\vartheta})|\, d\vartheta$$

is bounded as $r \to 1-0$. It is known that functions of the class $A$ have finite boundary values almost everywhere along nontangential paths, which are called their boundary values. If $w(\xi) \in A$ has zero boundary values on a set $E$ of positive measure on $|\xi| = 1$, then $w(\xi) \equiv 0$ in $|\xi| < 1$ (see, for example, ([1])).

Let us pose the following question: is it possible, by narrowing the class $A$, to achieve that an analogous situation holds on more sparse sets? Below a positive answer to this question is given.

2°. For functions $w(z)$ meromorphic in the disk $|z| < 1$, denote

$$T(r) = \int_0^r \frac{A(t)}{t}\, dt,$$

where $T(r)$ is the characteristic function in the sense of Nevanlinna, and $A(r)$ is the area of the domain of the Riemann surface onto which the function $w(z)$ maps the disk $|z| < r$. $A(r)$ is measured in the spherical metric.

In the work ([2]), M. M. Dzhrbashyan considered the classes $A(\alpha)$ of functions meromorphic in the unit disk whose characteristics satisfy the condition

$$\int_0^1 (1-r)^\alpha T(r)\, dr < \infty \qquad (\alpha > -1),$$

and obtained an integral representation of functions of these classes.

In particular, if a holomorphic function $w(z)$ from $A(\alpha)$ has no zeros, then it is represented in the form

$$w(z) = c_0 \exp\left[\frac{1}{\pi}\int_0^1\int_0^{2\pi} (1-\rho^2)^\alpha \frac{\lg|w(\rho e^{i\vartheta})|}{(1-z\rho e^{-i\vartheta})^{\alpha+2}}\, \rho\, d\rho\, d\vartheta\right]. \qquad (1)$$

Following L. Carleson ([3]), we introduce the following class of functions. We shall say that a function meromorphic in the disk $|z| < 1$ belongs to the class $T_\alpha$, $0 < \alpha \leq 1$, if

$$T_\alpha(w) \equiv \int_0^1 \frac{A(r)}{(1-r)^\alpha}\,dr < \infty \qquad \text{for } 0 < \alpha < 1,$$

$$T_1(w) \equiv \lim_{r\to 1-0} A(r) < \infty \qquad \text{for } \alpha = 1.$$

We note that since $T_\alpha \subset A(\alpha)$, for functions of the class $T_\alpha$ having no zeros, the representation (1) is valid.

For functions of the classes $T_\alpha$, L. Carleson proved ([3]) the following theorems.

A. $\lim\limits_{r\to 1-0} w(re^{i\vartheta})$ exists for all $\vartheta$, except, possibly, for a set whose outer $(1-\alpha)$-capacity is zero.

B. If $w(z) \not\equiv a$, then the outer $(1-\alpha)$-capacity of the set of values $\vartheta$ satisfying the equality $\lim\limits_{r\to 1-0} w(re^{i\vartheta}) = a$ is zero, with the possible exception of values $a$ belonging to a set of plane measure zero.

Since assertion B, generally speaking, does not hold for all $a$, it is impossible to obtain from it a uniqueness theorem in the classical form.

3°. We shall prove the following theorem.

**Theorem 1.** Let a function $w(z)$, holomorphic in $|z| < 1$, belong to the class $T_\alpha$, $0 < \alpha \leq 1$, and let $w(a_\nu) = 0$. If

$$\sum (1 - |a_\nu|)^{1-\alpha} < +\infty$$

and the limiting radial values of the function $w(z)$ are equal to zero on a set $E$, whose outer $(1-\alpha)$-capacity is positive, then $w(z) \equiv 0$ in $|z| < 1$.

**Proof.** Since $w(z) \in T_\alpha$, it can be represented in the form

$$w(z) = B(z) w_1(z), \tag{2}$$

where $B(z)$ is a Blaschke product, and the function $w_1(z)$ belongs to the class $T_\alpha$ ([3]).

Let $E \subset [0, 2\pi]$ be a certain set whose $(1-\alpha)$-capacity is positive. Then there is a unit distribution $\mu$ on $E$ such that

$$\int_0^{2\pi} \frac{d\mu(t)}{|e^{it} - re^{ix}|^{1-\alpha}} < c_1 \tag{3}$$

uniformly for all $x \in [0, 2\pi]$ and $r < 1$ ([3]).

We shall prove that

$$\int_0^{2\pi} \big|\lg|w(re^{i\vartheta})|\big| \, d\mu(\vartheta) < c_2, \tag{4}$$

where $c_2$ does not depend on $r$.

Denote by

$$B^{(r)}(z) = \prod_{|a_\nu|<r} \frac{a_\nu - z}{1 - \overline{a}_\nu z} \cdot \frac{\overline{a}_\nu}{|a_\nu|};$$

then, if $z = re^{i\vartheta}$, we have

$$\int_0^{2\pi} \big|\lg|B^{(r)}(z)|\big| \, d\mu(\vartheta) \leq \sum_{|a_\nu|<r} \int_0^{2\pi} \left|\lg\left|\frac{a_\nu - z}{1 - \overline{z} a_\nu} \cdot \frac{\overline{a}_\nu}{|a_\nu|}\right|\right| d\mu(\vartheta) \leq$$

$$\leq c_3 \sum_{|a_\nu|<r} (1-|a_\nu|) \int_0^{2\pi} \frac{d\mu(\vartheta)}{|z - a_\nu|} \leq c_3 \sum_{|a_\nu|<r} \frac{1 - |a_\nu|}{(r - |a_\nu|)^\alpha} \int_0^{2\pi} \frac{d\mu(\vartheta)}{|re^{i\vartheta} - |a_\nu| e^{i\vartheta_\nu}|^{1-\alpha}}.$$

Hence it is easy to see that

$$\int_0^{2\pi} |\lg|B(z)|| \, d\mu(\vartheta) \leq c_4 \sum_1^\infty (1 - |a_\nu|)^{1-\alpha} \int_0^{2\pi} \frac{d\mu(\vartheta)}{|1 - |a_\nu| e^{i(\vartheta - \vartheta_\nu)}|^{1-\alpha}} < c_5. \tag{5}$$

Now we use the integral representation (1) of the function $w_1(z)$. We have

$$\int_0^{2\pi} \big|\lg|w_1(re^{i\varphi})|\big| \, d\mu(\varphi) \leq$$

$$\leq c_6 + \frac{1}{\pi} \int_0^1 (1-\rho^2)^\alpha \rho \, d\rho \int_0^{2\pi} \big|\lg|w_1(\rho e^{i\vartheta})|\big| \, d\vartheta \int_0^{2\pi} \mathrm{Re} \frac{1}{(1 - z\rho e^{-i\vartheta})^{\alpha+2}} \, d\mu(\varphi) \tag{6}$$

or

$$\int_0^{2\pi} \big|\lg|w_1(re^{i\varphi})|\big| \, d\mu(\varphi) \leq c_6 + c_7 \int_0^1 (1-\rho^2)^\alpha \rho T_1(\rho) \, d\rho \int_0^{2\pi} \mathrm{Re} \frac{1}{(1 - z\rho e^{-i\vartheta})^{\alpha+2}} \, d\mu(\varphi).$$

Changing the order of integration and integrating by parts, we obtain

$$\int_0^{2\pi} \left|\lg |w_1(re^{i\varphi})|\right| \, d\mu(\varphi) \le$$

$$\le c_6 + c_8 \int_0^1 (1-\rho^2)^{\alpha+1} A_1(\rho)\, d\rho \int_0^{2\pi} \frac{d\mu(\varphi)}{\left|1 - r\rho e^{i(\varphi-\theta)}\right|^{1-\alpha} \left|1 - r\rho e^{i(\varphi-\theta)}\right|^{1+2\alpha}}. \quad (7)$$

From inequality (7) we have

$$\int_0^{2\pi} \left|\lg |w_1(re^{i\varphi})|\right| \, d\mu(\varphi) \le c_6 + c_9 \int_0^1 \frac{A_1(\rho)}{(1-\rho)^\alpha}\, d\rho \int_0^{2\pi} \frac{d\mu(\varphi)}{\left|1 - re^{i(\varphi-\theta)}\right|^{1-\alpha}} \le c_{10}. \quad (8)$$

From inequalities (5) and (8), taking (2) into account, we obtain (4). Applying Fatou' s lemma ([4]) in inequality (4), we obtain

$$\int_0^{2\pi} \left|\lg |w(e^{i\varphi})|\right| \, d\mu(\varphi) < +\infty.$$

It follows from this that on a set $E$ of positive $(1-\alpha)$-capacity, $f(e^{i\varphi})$ cannot be zero if $f(z) \not\equiv 0$. The theorem is proved.

Theorem 1 is easily extended to meromorphic functions of the class $T_\alpha$, if on the poles $b_\nu$ of the function $w(z)$ one imposes the condition

$$\sum_1^\infty (1 - |b_\nu|)^{1-\alpha} < +\infty.$$

4°. In ([5]) classes $T_H$ of meromorphic functions, analogous to the classes $T_\alpha$, are introduced as follows:

$w(z) \in T_H$, if

$$\int_0^1 H(1-r) A(r)\, dr < +\infty \qquad \text{when} \int_0^1 H(1-r)\, dr < \infty,$$

$$\lim_{r\to 1-0} A(r) < +\infty \qquad \text{when} \int_0^1 H(1-r)\, dr = \infty,$$

where $H$ satisfies fairly general conditions.

For the classes $T_H$, analogous results of L. Carleson are obtained ([5]) in terms of convex capacity ([6]).

We shall note only that if $\lambda_n$ is the convex capacity of the set $E$—positive

$$\left(\lambda_n = \sum_{k=n}^{\infty} \frac{1}{kH(1/k)}\right),$$

then, since, by Salem' s theorem ([7]), as $x \to 0$

$$\sum_1^{\infty} \lambda_n \cos nx \sim \int_1^{1/x} \frac{du}{H(1/u)} \geq \frac{1}{xH(x)} \geq \frac{c_{11}}{h(x)}, \qquad \text{where } h(x) = \int_0^x H(u)\, du,$$

there exists a unit distribution $\mu$ on $E$ such that

$$\int_0^{2\pi} \frac{d\mu(y)}{h(|x-y|)} < c_{12} \tag{9}$$

uniformly for $0 \leq x \leq 2\pi$.

If $w(z) \in T_H$, then it can be represented in the form

$$w(z) = B(z) w_1(z),$$

where $w_1(z)$ belongs to the class $T_H$ ([5]).

For the Blaschke product for which the series

$$\sum_1^{\infty} h(1 - |a_\nu|) < \infty$$

converges, a result analogous to (5) is obtained in the same way.

Let us further note that the representation of M. M. Dzhrbashyan remains valid for functions of the classes $T_H$. We shall use this representation [(8)]

$$w_1(z) = c_{13} \exp\left[\frac{1}{\pi} \int_0^1 \int_0^{2\pi} \lg |w_1(z)|\, G(z\bar{\zeta})\, h(1-\rho)\, d\rho\, d\vartheta\right], \tag{10}$$

where $z = re^{i\varphi}$, $\zeta = \rho e^{i\vartheta}$, and the function $G(z)$ has the form

$$G(z) = \sum_0^{\infty} \frac{z^n}{\alpha^n}, \qquad \text{where } \alpha_n = O\left(\frac{1}{n} h\left(\frac{1}{n}\right)\right).$$

For the distribution $\mu(\varphi)$ for which inequality (9) is satisfied, we obtain

$$\int_0^{2\pi} |\lg |w_1(re^{i\varphi})|| \, d\mu(\varphi) \leq$$

$$\leq c_{14} + c_{15} \int_0^1 h(1-\rho)\, d\rho \int_0^{2\pi} |\lg |w_1(\rho e^{i\vartheta})|| \, d\vartheta \int_0^{2\pi} \operatorname{Re} G(z\bar{\zeta})\, d\mu(\varphi).$$

Observing that from Salem’ s theorem [(7)] one can obtain the inequality

$$\sum_{n=0}^{\infty} \frac{n}{h(1/n)} \rho^n \cos nx \leq \frac{c_{16}}{h(x)(1-\rho)^2}$$

for the classes $T_H$, analogously to Theorem 1 we can prove the following theorem:

**Theorem 2.** *If an analytic function $w(z) \in T_H$ in the disk and the zeros $a_\nu$ of the function $w(z)$ satisfy the condition*

$$\sum h(1-|a_\nu|) < \infty,$$

*where*

$$h(t) = \int_0^t H(u)\, du,$$

*then from the fact that the function $w(z)$ has radial boundary values equal to zero on some set $E$ whose $\lambda_n$-capacity is positive,*

$$\left( \lambda_n = \sum_{k=n}^{\infty} \frac{1}{kH(1/k)} \right),$$

*it follows that $w(z) \equiv 0$ in $|z| < 1$.*

For meromorphic functions $w(z)$ of the class $T_H$ the analogous result is true if one requires that the poles $b_\nu$ of the function $w(z)$ satisfy the condition

$$\sum h(1-|b_\nu|) < +\infty.$$

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*Note: Figure translations are in progress. See original paper for figures.*