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Abstract

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BROWNIAN MOTION WITH REFLECTION AND THE PROBLEM OF THE INCLINED DERIVATIVE

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1. Let a continuous function $f(z)$ be given in the closure of a smooth plane domain D, and let a smooth vector field $l(z)$ be given on its boundary ∂D. For simplicity we shall assume that the vector $l(z)$ is tangent to ∂D only on a set M consisting of a finite number of points. We shall seek bounded functions $F(z)$ satisfying in D the equation $\Delta F = f$ (Δ is the Laplace operator), continuous outside the set M, and satisfying on $\partial D \setminus M$ the equation $dF/dl = 0$. This problem turns out to be closely connected with the study of the behavior of trajectories of a certain diffusion process*. In comparison with the well-studied (see [3,6]) problem of finding solutions continuous everywhere, called the problem with an inclined derivative, such a formulation of the problem has the advantage that it does not encounter topological difficulties. It is therefore useful to consider it as a first stage in the study of various generalizations of the problem with an inclined derivative, for example, the n-dimensional case of the problem. The subspace of continuous solutions of the problem is singled out from the space of bounded solutions by means of considerations of a topological character.

2. We make the following transformation of the vector field: $l_1(z) = l(z)$, if $(l(z), n(z)) \geq 0$, and $l_1(z) = -l(z)$ otherwise; here $n(z)$ denotes the vector of the inner normal, and the parentheses denote the scalar product. The vector field $l_1(z)$ may have singularities on the set M. We shall call a point $z \in M$ **positive** if $(z - y, l_1(y)) > 0$ for points $y \in \partial D,\ y \neq z$, sufficiently close to z. If, in some neighborhood of the point z, $(z-y, l_1(y)) < 0,\ y \neq z$, then we shall call the point z **negative**. Finally, if the scalar product $(z - y, l_1(y))$ assumes values of different signs in every neighborhood of the point $z \in M$, then we shall call z a **zero** point.

If n_+ (n_-) denotes the number of positive (respectively negative) points, and if the index n denotes the number of complete turns made by the vector field $l(z)$ in the negative direction when z traverses ∂D in the positive direction, then for a simply connected domain D the known index formula reads $n_+ - n_- = 2n + 2$.

3. Using the constructions of M. I. Freidlin ([1,6]), one can, by means of a limiting passage, construct a diffusion process X in the domain D with reflection along the field l_1 on $\partial D \setminus M$, stopped at the moment ζ of hitting the set M. Functions $f \in C^2(\overline{D} \setminus M)$ that belong to the domain of definition of the infinitesimal operator A (see ([2])) of this process satisfy on $\partial D \setminus M$ the condition $df/dl = 0$; on these functions $Af = \Delta f$.

Let a be a positive point. Denote by a_ε^+ (respectively a_ε^-) the intersection of the ε-neighborhood of the point a with the portion of the boundary ∂D lying in the positive (negative) direction of traversal from the point a (not including a). Denote by $p_\varepsilon^+(z)$ (respectively $p_\varepsilon^-(z)$) the probability that, starting from the point z, the trajectory of the process X hits the point a before exiting to a_ε^+ (a_ε^-). It is clear that as $\varepsilon \to 0$ the probabilities $p_\varepsilon^+(z)$ and $p_\varepsilon^-(z)$, increasing, tend to certain functions $p^+(z)$ and $p^-(z)$. It is natural to call the function

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* M. I. Freidlin drew my attention to the desirability of a probabilistic analysis of the problem with an inclined derivative.

$p^+(z)$ (respectively $p^-(z)$) the probability of entering the point z in the positive (negative) direction.

Denote by Γ_+ that part of the boundary ∂D near the zero point b such that $(y - b, l_1(y)) < 0$ for $y \in \Gamma_+$. Let $p_0(z)$ denote the probability that, starting from the point z, a trajectory of the process X will hit the point b.

Theorem 1. 1) The functions $p^+(z)$ and $p^-(z)$ are not identically equal to zero; moreover $p^+(z)$ ($p^-(z)$) tends to 1 when the point z approaches a, remaining on a_ε^- (respectively on a_ε^+). With probability 1, trajectories of the process can enter positive points either only in the positive direction or only in the negative direction. 2) The function $p_0(z)$ tends to 1 when z approaches the zero point, remaining on Γ_+. With probability 1, trajectories of the process X enter the zero point only from the side of Γ_+. 3) Finally, if positive or zero points exist, then the mean time $M_z\xi$ of absorption of the process X is bounded, and the probability of hitting a small neighborhood of a negative point C tends to 0 when the neighborhood shrinks to C.

We briefly indicate the proof of the theorem. The main difficulty is the estimation of absorption probabilities at special points. We describe the course of the argument for the case of a positive point a, since the other cases are treated analogously. Suppose first that, in some neighborhood G of the point a, the vector $l_0(z)$ is collinear with $z-a$ and tangent to ∂D only at the point a. Denote by Γ_ε the set of points z, $|z-a| \geq \varepsilon$, lying on $\partial D \cap \partial G$ in the negative direction of traversal from a, and by τ_ε the time at which the process X reaches $\partial G \setminus \Gamma_\varepsilon$.

If on $\partial G \setminus \Gamma_\varepsilon$ a bounded function $\varphi(z)$, continuous off a, is given, then the unique bounded function harmonic in G, continuous in $\overline{G} \setminus a$, equal to $\varphi(z)$ on $\partial G \setminus \Gamma_\varepsilon$, whose derivative along the field $l_0(z)$ on $\Gamma_\varepsilon \setminus \partial\Gamma_\varepsilon$ is equal to 0, is the mean value of the quantity $\varphi(z(\tau_\varepsilon))$, starting from z, $M_z\varphi(z(\tau_\varepsilon)) = \psi_\varepsilon(z)$ ($z(t)$ is the position of the process X at time t). If for $\varphi(z)$ one takes the argument of the angle formed by the vector $z-a$ with some fixed direction, then $\psi_\varepsilon(z) \equiv \varphi(z)$. From the fact that as $\varepsilon \to 0$ the function $\psi_\varepsilon(z)$ does not change, and from the boundedness, as $\varepsilon \to 0$, of the function $M_z\tau_\varepsilon$, one can derive that the probability $q(z)$ of absorption at a before the time τ_0 tends to 1 when the initial point z approaches a along Γ_0. Obviously, $p^+(z) \geq q(z)$, and $p^+(z) \to 1$ when $z \to a$ along Γ_0.

Let the vector $l(z)$ on Γ_0 form a smaller angle with the tangent vector at the point z, representing the positive direction of traversal, than $l_0(z)$. Then from the maximum principle it follows that the probability of absorption at a before exiting to $\partial G \setminus \Gamma_0$ for the process reflected along the field l is greater than $q(z)$. With the aid of an inversion transformation, any vector field in a sufficiently small neighborhood of a positive point can be transformed into a field possessing the property formulated above. But it is obvious that under inversion the functions of the form $\psi_\varepsilon(z)$ pass into similar functions for the transformed domain. Hence the results established for the special vector field are always valid.

The boundedness of $M_z\xi$ is proved according to the same plan. The mean value of the time needed by the process X to hit the union of a special point and the complement of its small neighborhood is first estimated for a special field l; then the results are transferred to the case of an arbitrary vector field and, finally, by ordinary methods it is proved that the mean of the quantity ξ itself is bounded. It is obvious that $p^+(z)$ and $p^-(z)$ are the probabilities of disjoint events. From Theorem 3 (see § 4) it follows that their sum is the probability of absorption at the given positive point, starting from z. Hence the second assertion of part 1) of Theorem 1 follows.

4. Number separately the positive and zero points, and denote by $p_i^+(z)$ ($p_i^-(z)$) the probability of hitting the i-th positive point

in the positive (negative) direction, and by $p_k^0(z)$ the probability of reaching the k-th, in order, zero point, starting from z.

We shall call an l-function a function $u(z)$ harmonic in D and satisfying, outside the singular points, the boundary condition $du/dl = 0$.

Theorem 2. The probabilities $p_j^+(z)$, $p_j^-(z)$, and $p_k^0(z)$ are l-functions u; conversely, every bounded l-function is a linear combination of these probabilities.

If $f(z) \in C(\overline{D})$ and $M_z\xi < \infty$, then the function

$$F(z) = M_z \left(\int_0^\xi f(z_t)\, dt \right)$$

satisfies, outside the singular points of the field l, the boundary condition $dF/dl = 0$, and inside the domain D

$$\Delta F = -f.$$

Theorem 3. Every l-function u and function $F(z)$ has limiting values when the argument tends along the boundary to a singular point in one of the directions of passage. If at some singular point the limiting values coincide, then the function is continuous in a two-dimensional neighborhood of this singular point.

Theorem 4. If an l-function $u(z)$ is continuous, then $u \in C^1(\overline{D})$ and satisfies everywhere on ∂D the boundary condition $du/dl = 0$.

It is not hard to prove that the extremum of an l-function is attained only at positive or zero singular points, and moreover at a zero point of extremum the l-function is discontinuous. Hence it follows easily that the space B of l-functions continuous at nonnegative points depends on n_+ parameters.

5. Denote by $\pi_k(z)^*$ that one of the l-functions continuous outside the negative points which is equal to 1 at the k-th positive point and to 0 at the remaining positive points. These functions form a basis of the space B. Denote by R_{jk} the difference of the limiting values of the function $\pi_k(z)$ as the point z tends along the boundary in the positive and negative directions to the j-th, in order, negative point. It is clear that the difference of the corresponding limiting values for the function $\sum a_k \pi_k(s)$ is $\sum a_{kR_{jk}}$. Thus, the problem of determining the space C of continuous l-functions is reduced to solving the system of equations

$$\sum_{k=1}^{n_+} a_k R_{jk} = 0; \qquad j = 1, 2, \dots, n_-.$$

We shall call two positive points **adjacent** if between them there are neither positive nor negative points. If D is simply connected and the index of the field l is equal to n, then there are $2n+1$ pairs of adjacent points. The subspace S of l-functions from B whose values at the points of these pairs coincide has dimension $n_+ - 2n - 1$. It is easy to prove that a nonconstant function from $S \cap C$ would have to have saddle points on the boundary between any two adjacent points. But, as is not difficult to extract from (5) or (4), it cannot have more than $2n$ saddle points. Thus $S \cap C$ is one-dimensional. From this the known result follows easily.

Theorem 5. If D is simply connected and $n \geq 0$, then

$$\dim C = n_+ - n_- = 2n + 2.$$

It is clear that the inhomogeneous problem $\Delta u = f,\ du/dl|_{\partial D} = 0,\ f \in C(\overline{D})$, under the conditions of Theorem 5, also has $2n+2$ linearly independent solutions $u \in C(\overline{D})$.

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* $\pi_k(z)$ is the probability of absorption at the k-th positive point of a process which differs from X in that, after arriving at a zero point, it leaves it.

Note: Figure translations are in progress. See original paper for figures.