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# Reports of the Academy of Sciences of the USSR

MATHEMATICS

1964

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## Abstract

## Full Text

Reports of the Academy of Sciences of the USSR  
1964. Volume 154, No. 1

MATHEMATICS

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# TENSOR FIELDS ON COMPACT HOMOGENEOUS SPACES ADMITTING A SPHERICAL MODEL

*(Presented by Academician I. G. Petrovskii, 18 VII 1963)*

Let there be given a homogeneous space  $K = G/H$ , where  $G \supset H$  are Lie groups,  $G$  is connected, and  $H$  is its closed subgroup. Suppose that we have constructed a faithful representation of the group  $G$  in the form of a group of motions  $\varphi(G)$  of the unit sphere  $S_n$  of the Euclidean space  $E_{n+1}$ , admitting such a closed surface of transitivity  $K^*$  that the stationary subgroup  $H$  of the group  $G$  is mapped into the stationary subgroup  $\varphi(H)$  of the group  $\varphi(G)$ , considered as a group of transformations on  $K^*$ , and at the same time the representation  $\overline{\varphi}(G)$  by a group of motions on  $K^*$  is faithful. In this case we shall say that  $K^*$  is a spherical model of the homogeneous space  $K$ .

It is clear that, in order for a homogeneous space  $K$  to admit a spherical model, it is necessary that the group  $G$ , and consequently also  $K$ , be compact. Conversely, in view of the compactness of  $G$ , it has a faithful representation by orthogonal matrices, and moreover the compact subgroup  $H$  is singled out algebraically in this representation; then one can construct a spherical model of the homogeneous space  $K$  in the same way as in <sup>(1)</sup> (p. 198) the projective model of the homogeneous space  $K$  is constructed.

In <sup>(2)</sup> a description was given of the spaces of tensor fields on the  $n$ -dimensional sphere  $S_n$ , irreducible with respect to the group of motions  $O(n+1)$  (which may be regarded as the homogeneous space  $K = O(n+1)/O(n)$ ).

The purpose of the present paper is, first, to give a scheme for obtaining analogous results in the general case, when  $K = G/H$  is a homogeneous space admitting a spherical model; and, second, to carry this out explicitly for the Grassmann manifold  $K = O(m)/O(m-k)O(k)$ .

The induced tensor field  $kT_{\alpha_1, \dots, \alpha_l}$ , introduced in <sup>(2)</sup>, may be interpreted as the projection onto the sphere  $S_n$  of a constant tensor field  $T_{i_1, \dots, i_k}$  of the space  $E_{n+1}$ , with the annihilated indices being projected onto the normal to  $S_n$ , and the remaining ones onto the plane tangent to  $S_n$ .

Let us take a spherical model  $K^*$  of our homogeneous space  $K$  on the unit sphere  $S_n$  of the Euclidean space  $E_{n+1}$ . We decompose the set of all possible tensors in  $E_{n+1}$  into linear spaces  $\Pi$ , irreducible with respect to the group  $\varphi(G)$ . Just as in <sup>(2)</sup>, we project, in all the indicated ways, these tensor spaces  $\Pi$ , irreducible with respect to  $\varphi(G)$ , onto the sphere  $S_n$ . We obtain, as in <sup>(2)</sup>, a complete system of spaces  $\Pi'$  of tensor fields on  $S_n$ , irreducible with respect to the group  $\varphi(G)$  (complete in the sense that every continuous tensor field on  $S_n$  admits uniform approximation by finite sums of our induced tensor fields).

Now we project the tensor fields obtained on  $S_n$  onto our model  $K^*$ . In doing so, we project all indices onto the plane tangent to  $K^*$ . The tensor fields on  $K^*$  obtained in this way will likewise be called induced.

Under the projection of an irreducible space  $\Pi'$  of tensor fields on  $S_n$  onto  $K^*$ , the preimage of the zero tensor field on  $K^*$  on  $S^*$  either fills the whole space  $\Pi'$ , or coincides with the zero tensor field on  $S_n$ . In the latter case the mapping under projection will be one-to-one, and the resulting space  $\Pi^*$  of tensor fields on  $K^*$  will be irreducible.

The set of irreducible spaces  $\Pi^*$  of induced tensor fields on  $K^*$  is complete in the sense that every continuous tensor field on  $K^*$  admits a uniform approximation by finite sums of the given induced tensor fields.

Indeed, every continuous tensor field  $P^*$  on  $K^*$  can be represented as the projection of a tensor field  $P'$ , continuous on  $S_n$ . If a sequence of tensor fields on  $S_n$  converges uniformly to the tensor field  $P'$ , then the sequence of projections on  $K^*$  of these tensor fields also converges uniformly to the projection  $P^*$  of the tensor field  $P'$ . Hence our assertion follows from Theorem 1 <sup>(2)</sup>.

With this we finish the consideration of the general case and pass to Grassmann manifolds.

Thus,

$$K = O(m)/O(m-k)O(k),$$

where  $O(m)$  is the proper orthogonal group.

As is known, the subgroup  $O(m-k)O(k)$  is singled out from the group  $O(m)$  by the condition of invariance of a simple  $k$ -vector (or of an  $(m-k)$ -vector orthogonal to it).

For  $E_{n+1}^1$ , where

$$n+1 = \frac{m!}{(m-k)!k!},$$

we take the Euclidean space of  $k$ -vectors  $z^{i_1, \dots, i_k}$  with metric

$$(z_i, z_j) = g_{i_1, j_1} g_{i_2, j_2} \dots g_{i_k, j_k} z^{i_1, i_2, \dots, i_k} z^{j_1, j_2, \dots, j_k}.$$

The space  $K^*$  will be the intersection of the manifold of simple  $k$ -vectors with the unit sphere  $S_n$ .

From a tensor  $T$  of valence  $ks$  of the space  $E_m$  we can obtain a tensor  $\hat{T}$  in the space  $E_{n+1}$  in the following way: the indices of the tensor  $T$  are divided into  $s$  groups of  $k$  indices in each, and alternation is carried out within each group; each such group is taken as an index in the space of  $k$ -vectors. We thus obtain in  $E_{n+1}$  a tensor  $\hat{T}$  of valence  $s$  (provided only that  $\hat{T} \neq 0$ ). Such a tensor in  $E_{n+1}$  will be called **induced**. It can be shown, relying on tensors having only one coordinate different from zero, that every tensor in  $E_{n+1}$  is representable in the form of a sum of induced tensors.

If, for tensors  $T$  in  $E_m$  belonging to an irreducible space  $\Pi$ , the indices are uniformly divided into groups, then the corresponding space  $\hat{\Pi}$  of tensors  $\hat{T}$  in  $E_{n+1}$  will be irreducible with respect to the group  $O(m)$ , acting in the obvious way in  $E_{n+1}$ . The correspondence between tensors of the spaces  $\Pi$  and  $\hat{\Pi}$  in each case will either be one-to-one, or the space  $\hat{\Pi}$  will be zero.

**Proposition 1.** *If  $m$  and  $k$  are even, then on  $K$  there acts not  $O(m)$ , but its factor by the symmetry about the origin, and in the space of tensor fields  $T^*$  on  $K^*$  there occur precisely those irreducible representations of the group  $O(m)$  which are equivalent to representations by means of irreducible tensors  $T$  in  $E_m$  with an even number of indices. If, however, at least one of the numbers  $m$  and  $k$  is odd, then every finite-dimensional irreducible representation of the group has an equivalent one among the irreducible representations in the space of tensor fields on*

$$K = O(m)/O(m-k)O(k).$$

The space of all tensors in  $E_m$  decomposes into subspaces irreducible with respect to the group  $O(m)$ , the tensors of each of which are representable in the form of a product of some number  $s$  of metric tensors by a traceless tensor with a Young scheme <sup>(3)</sup>.

For each such space the Young scheme and the number  $s$  of metric tensors are fixed. The set of such spaces  $\Pi$  of tensors  $T$  in  $E_m$  we shall call  $M$ . It is complete in the sense that every tensor in  $E_m$  is representable as a finite sum of tensors, each of which belongs to some irreducible space—an element of  $M$ . If, for each element of  $M$ , the indices of the tensors belonging to it are divided into groups of  $k$  indices in each (if this is possible) in all possible ways and, after alternating over these groups, one obtains spaces of tensors in  $E_{n+1}$  irreducible with respect to the group  $\varphi(O(m))$ , then the resulting set-

into  $M'$  of these spaces is complete in the preceding sense. In contrast to (3), we shall take the Young diagrams in skew-symmetric form.

It is of interest to note certain restrictions on the arrangement of groups of indices with respect to  $k$ , in spite of which the obtained set  $M'$  preserves completeness.

**Proposition 2.** *In order to preserve the completeness of the set  $M'$ , it is sufficient to take only those partitions into groups for which no group contains*

two indices from one row of the Young diagram or two indices of one metric tensor.

From the set  $M'$  we can, by the method described above, obtain a set  $M^*$  of spaces of tensor fields on  $K^*$ , irreducible with respect to the group  $\varphi(O(m))$ , complete in the sense that any continuous tensor field on  $K^*$  admits a uniform approximation by finite sums of continuous tensor fields, each of which belongs to an irreducible space—an element of  $M^*$ .

Moreover, it is of interest to consider also restrictions which, perhaps, destroy the completeness of  $M'$ , but preserve the completeness of  $M^*$ .

**Theorem.** *Without violating the completeness of the set  $M^*$ , among all possible ways of partitioning the indices of an element of  $\widehat{M}$  into groups one may take only those such that: 1) they satisfy Proposition 2; 2) each group, when projected (as an index of a tensor in  $E_{n+1}$ ) subsequently onto the normal to  $S_n$ , has no indices in rows of the Young diagram with number greater than  $k$ ; 3) each group of indices which, under projection onto  $S_n$ , is projected onto the tangent plane has no more than one index from rows of the Young diagram with number greater than  $k$ .*

It follows from this that the number of indices in rows with number greater than  $k$  must not exceed  $\frac{1}{k}$  of the total number of indices of the tensor.

The system of restrictions obtained is, unfortunately, not yet complete. For example, for  $K = O(m)/O(m-2)O(2)$  the tensor with Young diagram

$$\begin{array}{cc} 1 & 1 \\ 3 & 4 \\ 5 & 6 \end{array}$$

with groups of indices 1, 4; 3, 6; 2, 5 satisfies all the conditions, but already under the mapping into  $M'$  a zero element is obtained.

Received  
11 V 1963

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*Note: Figure translations are in progress. See original paper for figures.*

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