# MATHEMATICS

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## Abstract

## Full Text

# MATHEMATICS

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## On the Question of Riemannian Spaces with Reducible Isotropy Group

*(Presented by Academician A. N. Kolmogorov on 27 II 1964)*

**1.** Let a continuous group $G_r$ of motions act in a proper Riemannian space $V_n$, with a nontrivial reducible isotropy group $H$ (for all points of some domain). As is easy to see, one may always assume that the group $H$, acting in a proper Euclidean space, has no one-dimensional invariant directions in some invariant plane $E_{n-m}$. The plane $E_m$ orthogonal to $E_{n-m}$ is also invariant with respect to $H$, and thus $H = H_0 \times H_1$, where $H_0$ and $H_1$ denote the groups acting on the planes $E_m$ and $E_{n-m}$.

Choose a (generally speaking, nonholonomic) orthonormal frame $R$ so that its first $m$ vectors $e_\alpha{}^*$ belong to $E_m$, and the last $n-m$ vectors $e_a$ to $E_{n-m}$.

Every motion $\varphi \in G_r$ induces (at each point of $V_n$) an orthonormal frame $R^*$, whose vectors $e^*_\alpha, e^*_a$ also belong to the planes $E_m, E_{n-m}$.

Thus, if $p$ is the transition matrix from $R$ to $R^*$: $R^* = pR$, then

$$p^a_\alpha = p^\alpha_a = 0. \tag{1}$$

Let $\omega^i, \omega^i_j$ be the forms defining the connection of $V_n$ with respect to $R$; then for the quantities $\omega^{*i}, \omega^{*i}_j$, playing the analogous role with respect to $R^*$, one has

$$\omega^{*i} = q^i_j \omega^j, \tag{2a}$$

$$\omega^{*i}_j = q^i_s \omega^s_k p^k_j + q^i_s dp^s_j, \tag{2b}$$

where $q^i_s$ is the matrix inverse to $p^i_s$.

If $x'^i$ are the coordinates of the image of the point $M(x^k)$ under the motion $\varphi$, then, evidently,

$$\begin{aligned} \omega^{*i}(x', dx') &= \omega^i(x, dx), \\ \omega^{*i}_j(x', dx') &= \omega^i_j(x, dx). \end{aligned} \tag{I}$$

Now suppose that $\varphi$ belongs to the stationary subgroup of the point $M(x)$. If, moreover, it is assumed that the coordinates $x$ entering into (I) refer precisely to this point, then (I), with the aid of (2a) and (2b), can be rewritten in the form

$$q^i_j \omega^j(dx') = \omega^i(dx), \tag{3a}$$

$$\omega^i_k(dx')p^k_j + dp^i_j = p^i_k \omega^k_j(dx). \tag{3b}$$

Putting $i = a,\ j = \beta$, we obtain, in connection with (1),

$$\omega^a_\sigma(dx')p^\sigma_\beta = p^a_b \omega^b_\beta(dx). \tag{4}$$

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* Indices throughout range over the following limits: $\alpha, \beta, \gamma, \ldots = 1, \ldots, m;\quad a, b, c, \ldots = m+1, \ldots, n;\quad i, j, k = 1, \ldots, n.$

Representing $\omega^i_j$ in the form

$$\omega^i_j = \gamma^i_{jk}\omega^k,$$

where $\gamma^i_{jk}$ are functions of the coordinates, and replacing in (4), with the aid of (3a), the quantities $\omega^i(dx')$ by their expressions in terms of $\omega^k(dx)$, we obtain, after eliminating the arbitrary $\omega^k(dx)$,

$$\gamma^a_{\sigma k} p^k_i p^\sigma_\beta = p^a_b \gamma^b_{\beta i},$$

which is equivalent to two groups of equalities

$$\gamma^a_{\sigma b} p^b_c p^\sigma_\beta = p^a_b \gamma^b_{\beta c}, \tag{5a}$$

$$\gamma^a_{\sigma\tau} p^\tau_\alpha p^\sigma_\beta = p^a_b \gamma^b_{\beta\alpha}. \tag{5b}$$

**II. Case A.** Suppose first that $H_0$ (as well as $H_1$) admits no invariant one-dimensional directions. Then from (5a) and (5b) we immediately obtain

$$\gamma^a_{\sigma k} = 0 \quad \text{or} \quad \omega^a_\sigma = 0. \tag{6}$$

**Case B.** Now, without imposing any restrictions on $H_0$, suppose that the group $H_1$ (having no invariant one-dimensional directions) admits no nontrivial commutators. Then, putting in (5a), (5b) $p^\sigma_\beta = \delta^\sigma_\beta$, we obtain

$$\gamma^a_{\sigma b} = T_\sigma \delta^a_b, \qquad \gamma^a_{\sigma\beta} = 0,$$

where $T_\sigma$ are certain quantities, and $\delta^a_b$ is the Kronecker delta, or

$$\omega^a_\sigma = T_\sigma \omega^a. \tag{7}$$

We note that (6) is obtained from (7) when $T_\sigma = 0$.

From the conditions that the torsion be zero,

$$d\omega^i + \omega^i_j \wedge \omega^j = 0,$$

represented in the form of two groups of equalities

$$d\omega^a + \omega^a_\sigma \wedge \omega^\sigma + \omega^a_b \wedge \omega^b = 0, \tag{8a}$$

$$d\omega^\alpha + \omega^\alpha_b \wedge \omega^b + \omega^\alpha_\beta \wedge \omega^\beta = 0, \tag{8b}$$

it now follows, in view of (7) and $\omega^\sigma_a = -\omega^a_\sigma$, that each of the systems of Pfaffian equations $\omega^a = 0$, $\omega^\alpha = 0$ is completely integrable. Consequently, it is possible to choose local coordinates $x^i$ for which $\omega^a$ are linear combinations of the differentials $dx^b$, while $\omega^\alpha$ are differentials $dx^\beta$.

Introducing the symbols $d, \delta$ of differentiation with respect to the variables $x^\alpha, x^a$, respectively, which have the property

$$d\delta = \delta d = 0,$$

it is easy from (8a), (8b), taking (7) into account, to obtain

$$\delta \sum_a (\omega^a)^2 = 0, \tag{9a}$$

$$d \sum_a (\omega^a)^2 = 2T_\alpha \omega^\alpha \sum_a (\omega^a)^2. \tag{9b}$$

In case A the metric

$$ds^2 = \sum_\alpha (\omega^\alpha)^2 + \sum_a (\omega^a)^2,$$

computed for a special system of local coordinates, has, consetherefore, the form

$$ds^2 = g_{\alpha\beta}(x^\sigma)\, dx^\alpha dx^\beta + g_{ab}(x^c)\, dx^a dx^b$$

holds, and the following theorem is valid.

**Theorem 1.** *If the isotropy group $H$ of a proper Riemannian space $V_n$ is, for all points of some domain, the direct product of two groups $H_0$ and $H_1$, acting respectively in the orthogonal $G_r$-invariant planes $E_m$ and $E_{n-m}$ and admitting no one-dimensional fixed directions, then $V_n$ is reducible.*

Under the assumption that the group $G_r$ (for which $H$ is the isotropy group) is transitive, this theorem was proved by Wakakuwa ([1]).

In case B, as follows from (9a) and (9b), $ds^2$ has the form

$$ds^2 = g_{\alpha\beta}(x^\sigma)\, dx^\alpha dx^\beta + e^{2\theta(x^i)}\Pi_{ab}(x^c)\, dx^a dx^b, \tag{10}$$

where $\theta$ is some function of all the coordinates.

Starting from the fact that the curvature tensor of $V_n$ is preserved by the isotropy group, one can prove that (under a certain normalization) $\theta$ depends only on the coordinates $x^\sigma$.

The metric (10) (with $\theta = \theta(x^\sigma)$) characterizes a $V_n$ containing a geodesic field of directions ([2]); it has been considered by many authors and is known under the name semi-reducible ([3]).

In connection with the above, the following theorem may be formulated.

**Theorem 2.** *If the isotropy group $H$ of a proper Riemannian space is the direct product of groups $H_0$ and $H_1$, acting respectively in the $G_r$-invariant orthogonal planes $E_m$ and $E_{n-m}$, with $H_1$ admitting no invariant one-dimensional directions nor nontrivial commutators, then the metric of $V_n$ is semi-reducible.*

Under the assumption that the group $G_r$ is transitive, the main content of this theorem coincides with the theorem proved by G. I. Kruchkovich and Gu Chao-hao ([4]).

As noted in their work, the metric $\Pi_{ab}dx^a dx^b$ is (in the case of transitivity of $G_r$) Euclidean. If, however, transitivity of $G_r$ is not required, then *for any metric $\Pi_{ab}dx^a dx^b$ with an isotropy group admitting no one-dimensional invariant directions and no nontrivial commutators, the space with metric* (10) *admits a group of motions for which the conditions formulated in Theorem 2 hold.*

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*Note: Figure translations are in progress. See original paper for figures.*