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Abstract

Full Text

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On the Fisher–Welch–Wald Test

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In the present note we consider the question of the nonexistence of the Fisher—Welch—Wald test under certain assumptions on the test boundary. Let $x_1, x_2, \dots, x_{n_1} \in N(a_1, \sigma_1)$, $y_1, y_2, \dots, y_{n_2} \in N(a_2, \sigma_2)$ be normal repeated samples with different variances. The parameters $a_1, a_2, \sigma_1, \sigma_2$ are assumed unknown. The hypothesis $H_0 : a_1 = a_2$ is tested by means of a nonrandomized test similar with respect to the parameter $\theta = n_2\sigma_1^2/n_1\sigma_2^2$. The specification of the test determines a certain function $\psi(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2})$ $(0 \leq \psi \leq 1)$ —the probability of rejecting the hypothesis H_0 after the observations. If the function ψ takes only the two values 0 or 1, then the test is called **nonrandomized**; otherwise the test is **randomized**. Tests of this kind are completely characterized by specifying a critical region Z in the Cartesian product of the sample spaces. The test will be similar if the probability of falling into the critical region Z does not depend on the parameter θ. Using A. Wald' s conditions on the critical region Z, one can show $(^1)$ that the critical region has the form:

$$\frac{|\bar{x} - \bar{y}|}{\sqrt{s_1^2 + s_2^2}} \geq \varphi\left(\frac{s_1}{s_2}\right), \tag{1}$$

where $\bar{x}, \bar{y}, s_1^2, s_2^2$ are sufficient statistics for the four parameters $a_1, a_2, \sigma_1^2, \sigma_2^2$, and $\varphi(s_1/s_2)$ is a single-valued measurable function for $s_1/s_2 \geq 0$. If we put $(\bar{x} - \bar{y})/s_2 = \xi$, $s_1/s_2 = \eta$ and double the corresponding probabilities of falling into the region, then we obtain the region Z_1:

$$\xi \geq \Phi(\eta),$$

where ξ, η vary in the right upper quadrant $\Omega_1 : \xi \geq 0, \ \eta \geq 0$, and $\Phi(\eta) = \sqrt{1 + \eta^2}\,\varphi(\eta)$. The function $\Phi(\eta)$ has the same smoothness properties as $\varphi(\eta)$. Yu. V. Linnik $(^2)$, in the case of samples of equal size and under certain smoothness conditions on the function $\varphi(\eta)$, showed that there is no nonrandomized similar Fisher—Welch—Wald test. In the present note the results of $(^2)$ are somewhat strengthened, and they are extended to the case of samples of arbitrary size.

Considering the joint density of the distribution of the quantities ξ, η, and taking into account that the test must be similar, we arrive at the basic equation of the problem ([3]):

$$\iint_{\Omega_1} \psi(\xi,\eta)\,\frac{\eta^{n_1-2}\,d\xi\,d\eta}{\left[\theta^2+\theta(1+\xi^2+\eta^2)+\eta^2\right]^N} = C_{n_1,n_2}\,\frac{\alpha}{2}\,\theta^{-n_2/2}(1+\theta)^{-N+1/2}, \quad (2)$$

where $\theta = n_2\sigma_1^2/n_1\sigma_2^2$; C_{n_1,n_2} is a constant depending on n_1, n_2; $\alpha/2$ is the size of the critical region Z_1; $N = (n_1+n_2)/2 - 1/2$;

$$\psi(\xi,\eta) = \begin{cases} 1, & \text{if } (\xi,\eta)\in Z_1, \\ 0, & \text{if } (\xi,\eta)\notin Z_1. \end{cases}$$

The denominator of the fraction on the left in equation (2) vanishes only for negative values of θ. The zeros of the denominator generate

geometric loci of points, which are called **critics** ([3,4]). If in (2) $\theta = -D_0$ is a root of the denominator, then for $D \leqslant 1$ the critics $A(\xi,\eta) = D$ form a family of confocal hyperbolas of the form $\dfrac{\eta^2}{D} - \dfrac{\xi^2}{1-D} = 1$. If $D \geqslant 1$, then the critics $B(\xi,\eta) = D$ give a family of confocal semiellipses of the form $\dfrac{\xi^2}{D-1} + \dfrac{\eta^2}{D} = 1$.

Since both sides of (2) can be continued to the domain Λ of complex values of the parameter $\theta = \tau + i\zeta$ with a cut along the axis $\tau \leqslant 0$, and the integral on the left represents there an analytic function ([4]), choosing the corresponding branches of the factors for $\theta \in \Lambda$, we can rewrite the basic equation in the form

$$\iint_{\Omega_1} \psi(\xi,\eta)\,\frac{\eta^{n_1-2}\,d\xi\,d\eta}{(\theta+A(\xi,\eta))^N(\theta+B(\xi,\eta))^N} = C_{n_1,n_2}\,\frac{\alpha}{2}\,\theta^{-n_2/2}(1+\theta)^{-N+1/2}.$$

Theorem 1. There does not exist a similar nonrandomized test of A. Wald of the form (1), for which $\varphi(\eta)$ is continuous in the interval $(0,\eta_1]$, $\sup_{\eta\geqslant 1}\varphi(\eta) \leqslant c < \infty$; $\varphi(\eta)$ has a finite first derivative in the interval $[1+\varepsilon,\eta_1]$, where ε is a small number and η_1 is sufficiently large, and this function satisfies a Lipschitz condition in some interval.

In the theorem obtained in ([2]) for the case of samples of equal size, the condition is imposed that the function $\varphi(\eta)$ be continuous at the point $\eta = 0$. Here this condition is replaced by the weaker requirement that the function $\varphi(\eta)$ be bounded as $\eta \to +0$.

In the proof the following lemma is used essentially.

Lemma. Among the critic-hyperbolas there is a critic $A(\xi,\eta) = D_0$ ($D_0 \in (0,1)$), which is tangent to the test boundary $\Phi(\eta) = \xi$ at some point (ξ_0,η_0), with $\xi_0 \neq 0, \eta_0 > 1$.

The proof of the lemma is based on simple geometric considerations.

For the proof of the theorem one studies the behavior of the imaginary part of the basic equation when $\theta = -D_0 + i\zeta$ and $\zeta \to 0$ (D_0 is a root of the denominator). The domain of integration is divided into two parts: in one of them the denominator of the left-hand side of the basic equation is separated from zero by a certain constant. In this domain the imaginary part of the integral in (2), as $\zeta \to 0$, is a bounded quantity; while in the domain bounded by the two hyperbolas $A(\xi, \eta) = D_0 + \varepsilon_0$ and $A(\xi, \eta) = D_0 - \varepsilon_0$ (ε_0 sufficiently small) and containing the point of tangency of the critic $A(\xi, \eta) = D_0$ and the test boundary, the imaginary part of the integral under consideration tends to $-\infty$.

Consider a randomized test in which the function $\psi(\xi, \eta)$ represents the probability of accepting the hypothesis H_0. It is shown in (5) that randomized tests exist if the function $\psi(\xi, \eta)$ is discontinuous.

Theorem 2. If the function $\psi(\xi, \eta)$ increases monotonically in ξ for fixed η and is a discontinuous function, and moreover is such that the projection of the discontinuity onto the (ξ, η)-plane has the same smoothness properties in the interval $[1+\varepsilon, \eta_1]$ as the test boundary in Theorem 1, then $\psi(\xi, \eta)$ cannot define a randomized test similar with respect to the parameter θ.

Projecting the line of discontinuity of the function $\psi(\xi, \eta)$ onto the (ξ, η)-plane leads us to the conditions of Theorem 1.

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Note: Figure translations are in progress. See original paper for figures.