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Abstract

Full Text

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On the Principle of Limiting Amplitude

(Presented by Academician V. I. Smirnov on 27 IV 1964)

1. Let H be a Hilbert space; G a self-adjoint operator in H , $G \geq 0$; R_z the resolvent of the operator G ; H_0 a subspace in H ; P the projection operator onto H_0 . Let $w(t)$, where $0 < t < \infty$, be a function whose values are elements of H , having derivatives dw/dt , d^2w/dt^2 and satisfying the following equation and initial conditions:

$$\frac{d^2w}{dt^2} + Gw = fe^{-i\sqrt{\lambda}t}; \quad (1)$$

$$w|_{t=0} = 0, \quad \left. \frac{dw}{dt} \right|_{t=0} = 0, \quad (2)$$

where $\lambda < 0$, $f \in H$. The derivative dw/dt is assumed to be continuous for $0 \leq t < \infty$, and the derivative d^2w/dt^2 for $0 < t < \infty$. Continuity and differentiability are understood in the strong sense. It is known that problem (1), (2) is uniquely solvable.

Theorem 1. Let the element f be such that the following conditions are satisfied:

- 1) For every $\sigma > 0$ there exist the limits $PR_z f \rightarrow v^+(\sigma)$ as $z \rightarrow \sigma$, $\text{Im } z > 0$, and $PR_z f \rightarrow v^-(\sigma)$ as $z \rightarrow \sigma$, $\text{Im } z < 0$.
- 2) The function $\theta(\sigma) = v^+(\sigma) - v^-(\sigma)$ satisfies the Hölder condition on every interval $a \leq \sigma \leq b$, $a > 0$.

Then:

- a) If the function $\theta(\sigma)$ on some interval $0 < \sigma \leq \delta$ satisfies the inequality

$$\|\theta(\sigma)\| \leq c\sigma^{-\alpha}, \quad (3)$$

where $0 \leq \alpha < 1/2$, then for every $\lambda > 0$, as $t \rightarrow +\infty$,

$$e^{i\sqrt{\lambda}t}Pw \rightarrow v^+(\lambda).$$

- b) If on the interval $0 < \sigma \leq \delta$

$$\theta(\sigma) = \frac{\theta_0}{\sigma^\alpha} + \frac{\theta_1(\sigma)}{\sigma^\beta}, \quad (4)$$

where $\beta < 1/2$, $\alpha = 1/2$, $\|\theta_1(\sigma)\| \leq C$, then for every $\lambda > 0$, as $t \rightarrow +\infty$,

$$Pw - \left(e^{-i\sqrt{\lambda}t} v^+(\lambda) - \frac{\theta_0}{2\sqrt{\lambda}} \right) \rightarrow 0.$$

c) If on the interval $0 < \sigma \leq \delta$ (4) holds, where $1/2 < \alpha < 1$, $\beta < \alpha$, $\|\theta_1(\sigma)\| \leq C$, then for every $\lambda > 0$

$$Pw = t^{2\alpha-1}(\varkappa\theta_0 + \varphi(t)),$$

where $\varkappa = \varkappa_0/\sqrt{\lambda}$ (\varkappa_0 is an absolute constant), $\varphi(t) \rightarrow 0$ as $t \rightarrow +\infty$.

Moreover, if inequality (3) holds for some $\alpha < 1$, then for $s = 1, 2$ and $t \rightarrow +\infty$ one has

$$e^{i\sqrt{\lambda}t} P \frac{d^s w}{dt^s} \rightarrow (-i\sqrt{\lambda})^s v^+(\lambda).$$

2. Consider in an infinite domain Ω of n -dimensional space E_n the equation

$$\frac{\partial^2 w}{\partial t^2} + gw = e^{-i\sqrt{\lambda}t} f(x) \quad (5)$$

with zero initial and homogeneous boundary conditions. Here $\lambda > 0$, g is an elliptic differential operator of order $2m$. It is said that for equation (5) the **limiting-amplitude principle** holds if

$$\lim_{t \rightarrow +\infty} e^{i\sqrt{\lambda}t} w = v, \quad (6)$$

where v is a solution of the equation

$$gv = \lambda v + f. \quad (7)$$

A. N. Tikhonov and A. A. Samarskii, in the work (1), formulated this principle and proved that it holds in the case $\Omega = E_3$, $g = -\Delta$. O. A. Ladyzhenskaya, in the article (2), proved that (6) holds in the case $\Omega = E_3$, $g = -\Delta + q(x)$, $q(x)$ is finite and the operator g has no points of the discrete spectrum. G. I. Bass and A. G. Kostyuchenko (3) gave a proof in the case when $\Omega = E_n$, $2m < n$, g is a self-adjoint operator with constant coefficients.

In the present note the case of the operator

$$g = (-1)^m \sum_{|k|=|l|=m} D^k(a_k^l(x)D^l) + q(x),$$

is considered, where

$$k = (k_1, \dots, k_n), \quad |k| = \sum_{s=1}^n k_s, \quad D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}},$$

under the boundary conditions

$$\frac{\partial^p u}{\partial \nu^p} \Big|_{\Gamma} = 0, \quad (8)$$

where $p = 0, 1, \dots, m-1$; Γ is the boundary of the domain Ω ; ν is the normal to Γ . The boundary Γ (if it exists) of the infinite domain Ω will be assumed to consist of a finite number of closed nonintersecting $(n-1)$ -dimensional surfaces. Denote by S_ρ the sphere of radius ρ with center at the origin of coordinates, and by Ω_ρ and B_ρ the sets of all points of Ω lying respectively inside and outside S_ρ . Let $a_k^l = \overline{a_l^k}$, $q = \overline{q}$; $a_k^l = a_l^k$; $a_k^l(x) \in C^m(\overline{\Omega})$; $q(x)$ is measurable and bounded in Ω . Assume that for all real $\xi_k = \xi_{k_1 \dots k_n}$ and for all $x \in \Omega$ the inequality

$$\sum_{|k|=|l|=m} a_k^l(x) \xi_k \xi_l \geq \gamma \sum_{|k|=m} \xi_k^2$$

holds, where $\gamma > 0$ is a constant. Suppose that there exists $\rho_0 > 0$ such that in B_{ρ_0} $f(x) = 0$, $q(x) = 0$,

$$a_k^l(x) = \frac{m!}{k_1! \dots k_n!} \delta_{k_1}^{l_1} \dots \delta_{k_n}^{l_n}$$

(and thus in B_{ρ_0} , $g = (-\Delta)^m$). In addition, let $f \in L_2(\Omega)$. Below we shall assume that Γ lies inside S_{ρ_0} and $\rho > \rho_0$.

Let us introduce some classes of functions. Denote by $\overset{0}{D}_\rho$ the closure, in the metric of $W_2^{(m)}(\Omega_\rho)$, of the set of those functions each of which is defined in Ω_ρ and is identically zero in some strip along Γ . By $\overset{0}{D}$ denote the class of those functions which are defined in Ω and belong to $\overset{0}{D}_\rho$ for every $\rho > \rho_0$. Condition (8) will be understood in the sense $u \in \overset{0}{D}$. Denote by $\overline{W}_2^{(k)}$ the set of functions each of which is defined

in Ω and in any finite interior subdomain ω belongs to $W_2^{(k)}(\omega)$.

Put $\Phi = \overset{\circ}{D} \cap \overline{W}_2^{(2m)}$ and define the differential operator g on the set Φ . We shall say that a sequence $\varphi_k \in \Phi$ converges in the sense of Φ to a function φ , if $\varphi_k \rightarrow \varphi$ in any finite interior subdomain ω in the metric $W_2^{(2m)}(\omega)$ and, moreover, $\varphi_k \rightarrow \varphi$ in any Ω_ρ in the metric $W_2^{(2)}(\Omega_\rho)$. Then $\varphi \in \Phi$. Introduce the operator G on functions $u \in \Phi \cap L_2(\Omega)$, $gu \in L_2(\Omega)$, by means of the equality $Gu = gu$. Then G is a self-adjoint operator in $L_2(\Omega)$. Let R_z be the resolvent of the operator G . Put $u_z = R_z f$, where $\text{Im } z \neq 0$. Denote by A the set of all $\lambda > 0$ which are points of discontinuity of the spectral function of the operator G . Examples show that, generally speaking, $A \neq (0, \infty)$. However, in the case $g = (-\Delta)^m$ one has $A = (0, \infty)$.

Theorem 2. Let $\lambda \in A$, $\text{Im } z > 0$. Then, as $z \rightarrow \lambda$, $u_z \rightarrow u_\lambda^+$ in the sense of Φ , where u_λ^+ is the solution of equation (7) satisfying at infinity the radiation condition of I. N. Vekua (⁴).

If $\text{Im } z < 0$, then $u_z \rightarrow u_\lambda^-$ as $z \rightarrow \lambda$, where u_λ^- is the solution of equation (7) satisfying the conjugate radiation condition.

Theorem 3. Let $[a, b] \subset A$. Then for any $\rho > \rho_0$ there exists a constant c_ρ such that for any λ_1, λ_2 from $[a, b]$ the inequality holds

$$\|u_{\lambda_2}^\pm - u_{\lambda_1}^\pm\|_{L_2(\Omega_\rho)} \leq c_\rho |\lambda_2 - \lambda_1|.$$

Theorem 4. Let $q(x) \geq 0$ in Ω , and suppose at least one of the following conditions is fulfilled: 1) $2m < n$; 2) $q(x) > 0$ on a set of positive measure; 3) $\Omega \neq E_n$.

Then there exists a $\delta > 0$ such that $(0, \delta) \subset A$. Moreover, as $\lambda \rightarrow +0$, $u_\lambda^+ \rightarrow u_0$, $u_\lambda^- \rightarrow u_0$ in the sense of Φ , where u_0 is the solution of the equation

$$gu = f.$$

Let us note that u_0 satisfies the usual condition at infinity which is imposed for the polyharmonic equation.

Theorem 5. Let $\Omega = E_n$, $q(x) = 0$ in E_n , $2m \geq n$, $(0, \delta) \subset A$. Then:

1) for $2m = n$

$$u_\lambda^\pm = q^\pm \ln \frac{1}{\lambda} + \psi_\lambda^\pm;$$

2) for $2m - n = 1$

$$u_\lambda^\pm = q^\pm \lambda^{-1/2m} + \psi_\lambda^\pm;$$

3) for $2m - n \geq 2$

$$u_\lambda^\pm = q^\pm \lambda^{n/2m-1} + \psi_\lambda^\pm \lambda^{(n+2)/2m-1},$$

where

$$\|\psi_\lambda^\pm\|_{L_2(\Omega_\rho)} \leq C(\rho)$$

for $0 < \lambda < \delta$; $q^\pm = \tau^\pm \int_\Omega f dx$; τ^+ , τ^- are constants depending only on m, n , and $\tau^+ \neq \tau^-$.

We now consider the solution $w(t)$ of problem (5) with zero initial conditions and the homogeneous boundary conditions introduced above.

Theorem 6. Let $A = (0, \infty)$, $q(x) \geq 0$ in Ω , and suppose at least one of the following conditions is fulfilled: 1) $m < n$; 2) $q(x) > 0$ on a set of positive measure; 3) $\Omega \neq E_n$.

Then for any $\lambda > 0$, as $t \rightarrow +\infty$,

$$e^{i\sqrt{\lambda}t}w \rightarrow u_\lambda^+$$

in the sense of Φ . Moreover, in any domain Ω_ρ , as $t \rightarrow +\infty$,

$$e^{i\sqrt{\lambda}t} \frac{d^s w}{dt^s} \rightarrow (i\sqrt{\lambda})^s u_\lambda^+ \quad (9)$$

in the sense of $\mathcal{L}_2(\Omega_\rho)$, where $s = 1, 2$.

Theorem 7. Let $A = (0, \infty)$, $\Omega = E_n$, $q(x) = 0$ in E_n . Then:

1) If $m = n$, then for any $\lambda > 0$

$$w(t) = e^{-i\sqrt{\lambda}t} u_\lambda^+ + \frac{\mu}{\sqrt{\lambda}} \int_\Omega f dx + \varphi(t).$$

2) If $m > n$, then for any $\lambda > 0$

$$w(t) = t^{1-n/m} \left(\frac{\mu}{\sqrt{\lambda}} \int_\Omega f dx + \varphi(t) \right),$$

where $\|\varphi(t)\|_{\mathcal{L}_2(\Omega_\rho)} \rightarrow 0$ as $t \rightarrow +\infty$; μ is a constant depending only on m, n ; $\mu \neq 0$. Moreover, for $m \geq n$, (9) holds.

Theorems 6 and 7 are proved with the aid of Theorem 1, whose conditions turn out to be satisfied by virtue of Theorems 2, 3, 4, 5. Here the role of the space H is played by the space $\mathcal{L}_2(\Omega)$, and the role of the subspace H_0 by the set of functions from $\mathcal{L}_2(\Omega)$ identically equal to zero in B_ρ .

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