Reports of the Academy of Sciences of the USSR

MATHEMATICS

1964

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1964. Vol. 158, No. 5

MATHEMATICS

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ON THE MAGNITUDES OF THE DEFECTS OF A MEROMORPHIC FUNCTION

(Presented by Academician M. A. Lavrent' ev on 15 VI 1964)

§ 1. Let $f(z)$ be a function meromorphic in the open plane; $T(r, f)$, $m(r, a)$, $n(r, a)$, $N(r, a)$, $\delta(a, f)$ are the quantities introduced by R. Nevanlinna, characterizing the distribution of the values of this function. We agree to denote

$$n(r) = n(r, 0) + n(r, \infty), \quad N(r) = N(r, 0) + N(r, \infty),$$

by the letter K with indices absolute constants, and by the letter C with indices quantities depending only on the function under consideration.

As is known (^{1}), the defect quantities satisfy the relation

$$\sum_{(a)} \delta(a) \leqslant 2$$

(the sum is extended over all values a with $\delta(a, f) > 0$). Put

$$s(f) = \sum_{(a)} \sqrt{\delta(a)}, \quad \sigma(\lambda) = \sup_{(f)} s(f),$$

where the supremum is taken over all meromorphic functions of lower order λ.

B. Fuchs (^{2}) proved that for $\lambda < \infty$ the quantity $\sigma(\lambda)$ is finite and that the estimate

$$\sigma(\lambda) \leqslant K_1 \left(1 + \sqrt{\lambda\,|\ln \lambda|}\right). \tag{1,1}$$

is valid.

The main result of this article is the following theorem.

Theorem 1. The inequality

$$\sigma(\lambda) \leqslant K_2 \sqrt{\lambda}, \quad 0.5 \leqslant \lambda < \infty. \tag{1,2}$$

is valid.

For the entire function (([1], p.240)

$$h_p = \int_0^z e^{t^p}\, dt, \quad p = 1, 2, 3, \dots,$$

one has $\lambda = p$,

$$s(h_p) = 1 + \sqrt{p} = 1 + \sqrt{\lambda}.$$

Therefore estimate (1,2) is sharp for large λ in the sense of order.

We obtain Theorem 1 as a consequence of two theorems: Theorem 2, due to Fuchs ([2]), and Theorem 3, established by us.

Theorem 2 ([2]). If $f(z)$ is a meromorphic function of finite lower order λ and has at least two deficient values, then the relation

$$\sum_{(a)} \sqrt{\delta(a)} \leqslant \left\{ 2\pi \lim_{r\to\infty} [T(r,f)]^{-1} r\, \mathfrak{M}\left(r, \frac{f''}{f'}\right) \right\}^{1/2}, \tag{1,3}$$

is valid, where

$$\mathfrak{M}(r,g) = \frac{1}{2\pi} \int_0^{2\pi} \left| g\left(re^{i\theta}\right) \right| d\theta.$$

Theorem 3. If $f(z)$ is a meromorphic function of lower order λ ($\lambda \geqslant 0.5$), then

$$\lim_{r\to\infty} [T(r,f)]^{-1} r\, \mathfrak{M}\left(r, \frac{f''}{f'}\right) \leqslant K_3 \lambda. \tag{1,4}$$

§ 2. **Auxiliary propositions.**

Lemma 1. Let $f(z)$ be a function meromorphic in the sector

$$G_{a,R,\vartheta} = \{z : \ 0 < |z| < R, \ |\arg z - \vartheta| < a\}.$$

For any real ϑ, any α ($0 < \alpha < \pi$), and r ($0 < r_0 < r < 0.5R$), the inequality holds

$$\begin{aligned}\frac{r}{2\pi}\int_{-\alpha/2}^{\alpha/2}\left|\frac{f'\left(re^{i(\varphi+\vartheta)}\right)}{f\left(re^{i(\varphi+\vartheta)}\right)}\right| d\varphi \leq \alpha^{-1}\int_{r_0}^{R} & \left\{\left|\ln\left|f\left(te^{i(\vartheta+\alpha)}\right)\right|\right| + \left|\ln\left|f\left(te^{i(\vartheta-\alpha)}\right)\right|\right|\right\} P(t,r,\alpha)\, dt \\ & + K_4\alpha^{-1}\int_0^R \left\{\left|\ln\left|f\left(te^{i(\vartheta+\alpha)}\right)\right|\right| + \left|\ln\left|f\left(te^{i(\vartheta-\alpha)}\right)\right|\right|\right\}\left(\frac{r}{R^2}\right)^{x} t^{x-1}\, dt \\ & + K_5\alpha^{-1}\int_{-\alpha}^{\alpha}\left|\ln\left|f\left(Re^{i(\vartheta+\theta)}\right)\right|\right|\left(\frac{r}{R}\right)^{x} d\theta \\ & + 2\sum_{c_m\in G_{\alpha,R,0}}\left(\frac{r}{|c_m|}\right)^{x}\Phi\left[\left(\frac{r}{|c_m|}\right)^{x}\right] \\ & + K_6 \sideset{}{'}\sum_{c_m\in G_{\alpha,R,0}}\left(\frac{r}{R^2}\right)^{x}|c_m|^{x} + C_1,\end{aligned} \tag{2.1}$$

where

$$x = x(\alpha) = \pi(2\alpha)^{-1}, \qquad P(t,r,\alpha) = t^{x-1}r^{x}(t^{x}+r^{x})^{-1}, \qquad \Phi(u) = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|ue^{i\theta}-1|},$$

$c_m = c_m(\vartheta)$ are the zeros and poles of the meromorphic function $f(ze^{i\vartheta})$.

The proof is based on the representation of $\ln f(re^{i\theta})$ in the sector $G_{\alpha,R,0}$, analogous to the Schwarz-Nevanlinna formula ((1), p. 165).

Lemma 2. Let $f(z)$ be a function meromorphic in $|z| \leq R < \infty$. For $0 < r_0 < r < 0.5R$, $0 < \alpha < \pi$, the estimate holds

$$\begin{aligned}r\mathfrak{m}\left(r,\frac{f'}{f}\right) \leq K_7\alpha^{-2} & \int_{r_0}^{R}\{m(t,0)+m(t,\infty)\}P(t,r,\alpha)\, dt \\ & + K_8\alpha^{-1}\left(\frac{r}{R}\right)^{x} T(R,f) + K_9\sum_{|c_m|<R}\left(\frac{r}{|c_m|}\right)^{x}\Phi\left[\left(\frac{r}{|c_m|}\right)^{x}\right] \\ & + K_{10}\left(\frac{r}{R}\right)^{x} n(R) + C_2\alpha^{-1}.\end{aligned} \tag{2.2}$$

Proof. Let $\vartheta_k = \beta + k\alpha$, where $0 \leq \beta < 2\pi$, and k takes the values $0, 1, \dots, q = [4x]$. Putting $\vartheta = \vartheta_k$ $(k = 0, 1, \dots, q)$ in inequality (2.1), we obtain $q+1$ inequalities. Adding these inequalities over k from 0 to q, we shall have

$$
\begin{aligned}
r\mathfrak{m}\left(r, \frac{f'}{f}\right) \leq \alpha^{-1} \sum_{k=0}^{q} \int_{r_0}^{R} & \left\{\left|\ln\left|f\left(te^{i(\beta+(k+1)\alpha)}\right)\right|\right| + \left|\ln\left|f\left(te^{i(\beta+(k-1)\alpha)}\right)\right|\right|\right\} P(t, r, \alpha)\, dt \\
& + K_4 \alpha^{-1} \sum_{k=0}^{q} \left\{\left|\ln\left|f\left(te^{i(\beta+(k+1)\alpha)}\right)\right|\right| + \left|\ln\left|f\left(te^{i(\beta+(k-1)\alpha)}\right)\right|\right|\right\} \left(\frac{r}{R^2}\right)^{x} t^{x-1}\, dt \\
& + K_5 \alpha^{-1} \sum_{k=0}^{q} \int_{-\alpha}^{\alpha} \left|\ln\left|f\left(Re^{i(\theta+\beta+k\alpha)}\right)\right|\right| \left(\frac{r}{R}\right)^{x} d\theta \\
& + 2 \sum_{k=0}^{q} \sum_{c_m(\vartheta_k) \in G_{\alpha,R,0}} \left(\frac{r}{|c_m|}\right)^{x} \Phi\left[\left(\frac{r}{|c_m|}\right)^{x}\right] \\
& + K_6 \sum_{k=0}^{q} \sum_{c_m(\vartheta_k) \in G_{\alpha,R,0}} \left(\frac{r}{R^2}\right)^{x} |c_m|^{x} C_2 \alpha^{-1}.
\end{aligned}
\tag{2.3}
$$

Obviously, the inequalities hold

$$
2 \sum_{k=0}^{q} \sum_{c_m(\vartheta_k) \in G_{\alpha,R,0}} \left(\frac{r}{|c_m|}\right)^{x} \Phi\left[\left(\frac{r}{|c_m|}\right)^{x}\right] \leq K_9 \sum_{|c_m| \leq R} \left(\frac{r}{|c_m|}\right)^{x} \Phi\left[\left(\frac{r}{|c_m|}\right)^{x}\right],
\tag{2.4}
$$

$$
\sum_{k=0}^{q} \int_{-\alpha}^{\alpha} \left|\ln\left|f\left(Re^{i(\theta+\beta+k\alpha)}\right)\right|\right| d\theta \leqslant K_8 T(R, f)
\tag{2,5}
$$

$$
\sum_{k=0}^{q} \sum_{c_m(\vartheta_k) \in G_{\alpha,R,0}} \left(\frac{r}{R^2}\right)^{\chi} |c_m|^{\chi} \leqslant K_{11} \sum_{|c_m| \leq R} \left(\frac{r}{R^2}\right)^{\chi} |c_m|^{\chi}.
\tag{2,6}
$$

Replacing in inequality (2,3) the expressions occurring on the left-hand sides of (2,4), (2,5), and (2,6) by the expressions occurring on the right-hand sides, and integrating the resulting inequality with respect to β from 0 to 2π, we obtain inequality (2,2).

§ 3. **Proof of Theorem 3.** Let $f(z)$ have lower order λ and order ρ. We shall carry out the proof under the assumption that $\lambda < \rho$. Choose γ so that $\lambda < \gamma < \rho$, and take $\alpha < \pi(2\gamma)^{-1}$. Divide inequality (2,2) by $r^{\gamma+1}$ and integrate it with respect to r from r_0 to $0.5R$; we obtain

$$\int_{r_0}^{0.5R} r^{-\gamma-1}\left\{r\mathfrak{M}\left(r,\frac{f'}{f}\right)\right\}dr \leqslant K_7\alpha^{-2}\int_{r_0}^{0.5R}\{m(t,0)+m(t,\infty)\}\times$$
$$\times\int_{r_0}^{0.5R} r^{-\gamma-1}P(t,r,\alpha)\,dr\,dt + K_7\alpha^{-2}\int_{0.5R}^{R}\{m(t,0)+m(t,\infty)\}\times$$
$$\times\int_{r_0}^{0.5R} r^{-\gamma-1}P(t,r,\alpha)\,dr\,dt + K_{11}(\pi-2\alpha\gamma)^{-1}R^{-\gamma}T(R,t)+$$
$$+K_9\sum_{r_0<|c_m|<0.5R}\int_{r_0}^{0.5R} r^{-\gamma-1}\left\{\left(\frac{r}{|c_m|}\right)^{\chi}\Phi\left[\left(\frac{r}{|c_m|}\right)^{\chi}\right]\right\}dr$$
$$+K_{12}(\pi-2\alpha\gamma)^{-1}R^{-\gamma}n(R)+$$
$$+K_9\sum_{0.5R\leq|c_m|<R}\int_{r_0}^{0.5R} r^{-\gamma-1}\left\{\left(\frac{r}{|c_m|}\right)^{\chi}\Phi\left[\left(\frac{r}{|c_m|}\right)^{\chi}\right]\right\}dr + C_4\alpha^{-1}. \tag{3,1}$$

Using the relations (see ([10], [3]))

$$\int_0^{\infty} r^{-\gamma-1}P(t,r,\alpha)\,dr = t^{-\gamma-1}\alpha\sec\alpha\gamma, \qquad \int_0^{\infty} r^{-\sigma}\Phi(r)\,dr \leqslant 4.4\operatorname{cosec}\pi\sigma \quad (0<\sigma<1), \tag{3,2}$$

from (3,1) we obtain

$$\sin 2\alpha\gamma\int_{r_0}^{0.5R} r^{-\gamma-1}\left\{r\mathfrak{M}\left(r,\frac{f'}{f}\right)\right\}dr \leqslant 2K_7\alpha^{-1}\sin\alpha\gamma\int_{r_0}^{0.5R} r^{-\gamma-1}\{m(r,0)+m(r,\infty)\,dr\}+$$
$$+2K_{13}\alpha^{-1}\sin\alpha\gamma\,R^{-\gamma}T(R,t)+K_{11}(\pi-2\alpha\gamma)^{-1}\sin 2\alpha\gamma\,R^{-\gamma}T(R,f)+$$
$$+K_{14}\alpha\gamma^2\int_{r_0}^{0.5R} r^{-\gamma-1}N(r)\,dr + C_5R^{-\gamma}T(2R,f)+C_6. \tag{3,3}$$

Choose in this inequality $\alpha=\pi(4\gamma)^{-1}$. Applying it then to $f'(z)$ instead of $f(z)$ (this can be done, since the order and lower order of $f(z)$ and $f'(z)$ coincide) (([4]), p. 52) and taking into account the relation (([5]), p. 61)

$$T(r,f') \leqslant 2T(r,f)+4\ln^+T(2r,f)+4\ln^+r+K_{15} \qquad (0<r_0<r),$$

from (3,3) we find

$$\int_{r_0}^{0.5R} r^{-\gamma-1}\left\{r\mathfrak{M}\left(r,\frac{f''}{f'}\right)\right\}dr \leqslant K_{16}\gamma\int_{r_0}^{0.5R} r^{-\gamma-1}T(r,f)\,dr+$$

$$+C_7 \int_{r_0}^{0.5R} r^{-\gamma-1} \ln^+ T(r, f)\, dr + C_8 R^{-\gamma}\{T(4R, f) + \ln^+ T(4R, f)\} + C_8.$$

From this inequality, by means of arguments analogous to those used in (^{9}, ^{7}, ^{10}), we obtain

$$\varliminf_{r\to\infty} [T(r, f)]^{-1} \left\{ r\mathfrak{M}\left(r, \frac{f''}{f'}\right)\right\} \leqslant K_{16}\gamma.$$

Letting now γ tend to λ, we obtain the assertion of the theorem.

For $\lambda = \rho$ the proof of Theorem 3 is carried out as follows. Divide inequality (2.2) with $\alpha = \pi(4\rho)^{-1}$ by $r^{\rho(r)+1}$, where $\rho(r)$ is the refined order of the meromorphic function $f(z)$, and integrate with respect to r from r_0 to $0.5R$. Arguing further analogously to the case $\lambda < \rho$, but in estimating the integrals (3.2) using Lemma 2 from (^{6}) (p. 78), we arrive at the relation

$$\int_{r_0}^{0.5R} r^{-\rho(r)-1} \left\{ r\mathfrak{M}\left(r, \frac{f''}{f'}\right)\right\} dr \leqslant K_{18}\rho \int_{r_0}^{0.5R} r^{-\rho(r)-1} T(r, f)\, dr +$$

$$+ C_{10} \int_{r_0}^{0.5R} r^{-\rho(r)-1} \ln^+ T(r, f)\, dr + C_{11} R^{-\rho(R)}\{T(4R, f) + \ln^+ T(4R, f)\} + C_{12}.$$

Taking into account the properties of the refined order, we have

$$\lim_{R\to\infty} R^{-\rho(R)} T(R, f) = 1, \qquad \lim_{R\to\infty} \int_{r_0}^{0.5R} r^{-\rho(r)-1} T(r, f)\, dr = \infty.$$

Using these relations and arguing analogously (cf. (^{9}, ^{7}, ^{10})) to the case $\lambda < \rho$, we obtain the assertion of the theorem.

§ 4. **Theorem 4.** Let Δ be the sum of the deficiencies of a meromorphic function $f(z)$ of finite lower order λ. Then $f(z)$ has at least one deficiency satisfying the condition

$$\delta(a) > \Delta^2 (4K_2^2 \lambda)^{-1}.$$

For the proof we use Theorem 1 and arguments analogous to (^{2}) (p. 209).

§ 5. **Remark.** In (^{7}) we obtained the following result, supplementing Theorem 1.

Theorem 5. The estimate

$$\sigma_1(\lambda) \leqslant K_{19}\sqrt{\lambda}, \qquad 0 < \lambda < 0.5, \tag{5.1}$$

is valid, where $\sigma_1(\lambda) = \sup_{(f)} S(f)$, the supremum being taken over all meromorphic functions of lower order λ having at least two deficient values.

With the aid of the method of (^{2}), from this theorem we obtain the following result.

Theorem 6. If a meromorphic function $f(z)$ of lower order λ ($\lambda < 0.5$) has at least two deficient values, then

$$\sum_{(a)} \delta(a) \leqslant K_{20}\lambda^{3/2}. \tag{5.2}$$

This theorem strengthens the result of Edrei and Fuchs (^{8}). We have not been able to establish the sharpness of the estimates (5.1) and (5.2). We suppose that in (5.2) the exponent 3/2 can be replaced by 2.

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Received
8 VI 1964

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